MIDTERM I STUDY GUIDE

Our midterm exam will take place on Friday, 05 March 2021, during class in our usual classroom Johnson Hall 415. It will be 50 minutes long, starting promptly at 11:00 am.

The exam will be about the material covered in class from sections 1.1–1.4, 2.1–2.5, and 3.1–3.3 from the course textbook by R. Haberman. You should be comfortable with all the homework problems and the examples from class. I strongly recommend to work through the practice problems below as part of your preparations for the midterm exam (I will soon upload solutions to these practice problems). The exam will consist of three problems whose difficulty will be similar to the difficulty of the practice problems below.

Over the last few weeks you have aqcuired the following skills that will be key for solving the exam problems. Please view your preparation for our first midterm exam as a great opportunity to consolidate your knowledge of these important techniques for solving linear PDEs:

- use the method of separation of variables to solve the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ on an interval subject to different boundary conditions: zero boundary conditions (Dirichlet), insulated boundary conditions (Neumann), periodic boundary conditions,
- use the method of separation of variables to solve Laplace's equation $\Delta u = 0$ inside a rectangle, inside a disk, or inside a quarter-circle,
- sketch the graph of the Fourier series, the Fourier Cosine series, or the Fourier Sine series of a given function over an interval.

You will *not* be allowed to use any textbooks, notes, calculators, cellphones or other devices in the exam. In order to receive full credit on your solutions to the exam problems, please make sure to *justify* your answers. You are free to use results from class or from the course textbook as long as you clearly state what you are using.

I will hold an extended office hour from 3 pm to 5 pm on Wednesday, March 3th . If you have any questions or concerns, please come to see me after class or in my office hours or send me an email.

Good luck and all the best for the exam!
Ahmed

Problem 1. Solve the following linear heat equation

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad -1 \leq x \leq 1,\, t > 0,\\ u(-1,t) &= u(1,t),\\ \frac{\partial u}{\partial x}(-1,t) &= \frac{\partial u}{\partial x}(1,t),\\ u(x,0) &= 4\cos(10\pi x) - \sin(5\pi x).\end{aligned}$$

Problem 2. Determine all eigenvalues of the following boundary value problem

$$\begin{aligned}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} &= -\lambda\phi, \quad 0 \leq x \leq \frac{\pi}{2},\\ \phi(0) &= 0,\\ \frac{\mathrm{d}\phi}{\mathrm{d}x}(\frac{\pi}{2}) &= 0.\end{aligned}$$

Problem 3.

(a) Consider the function

$$f(x) = \begin{cases} -x, & -1 \leq x < 0, \\ -1 + x, & 0 \leq x \leq 1. \end{cases}$$

Sketch the Fourier series of $f(x)$ over the interval $-1 \leq x \leq 3$.

(b) Consider the function

$$g(x) = x^2, \quad 0 \leq x \leq 1.$$

Sketch the Fourier sine series of $g(x)$ and the Fourier cosine series of $g(x)$ over the interval $-1 \leq x \leq 3$.

Problem 4. Solve the following Laplace's equation inside a rectangle

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad 0 \leq x \leq \pi,\, 0 \leq y \leq \pi\\ u(x,0) &= 0\\ u(x,\pi) &= 5\sin(3x) + 10\sin(7x)\\ u(0,y) &= 0\\ u(\pi,y) &= 0\end{aligned}$$

Problem 5. Solve the following Laplace's equation inside the quarter-circle of radius 1

$$\begin{aligned}\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0, \quad 0 \leq r \leq 1,\, 0 \leq \theta \leq \frac{\pi}{2}\\ u(r,0) &= 0\\ u(r,\tfrac{\pi}{2}) &= 0\\ u(1,\theta) &= \sin(8\theta)\end{aligned}$$

Hint: Recall that if one plugs the product ansatz $u(r,\theta) = \phi(\theta)G(r)$ *into* $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, *then one obtains that* $\frac{r}{G}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}G}{\mathrm{d}r}\right) = -\frac{1}{\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}\theta^2}$. *The ODE* $r^2\frac{\mathrm{d}G}{\mathrm{d}r} + r\frac{\mathrm{d}G}{\mathrm{d}r} - n^2 G = 0$ *has the general solution* $G(r) = c_1 r^{+n} + c_2 r^{-n}$ *for* $n \neq 0$ *and* $G(r) = \overline{c}_1 + \overline{c}_2\ln(r)$ *for* $n = 0$.