MATH 5410 LECTURE NOTES
THE FOURIER TRANSFORM

Topics covered

- Complex Fourier series
- Fourier transform
  - Extending Fourier series to infinite intervals
  - Derivatives and LCC operators
  - Gaussian transform
  - Convolutions
- Use in solving DEs
  - Solving LCC ODEs: symbol; Green's function
  - The heat equation; fundamental solution
  - Convolutions: interpreting the solution
  - Limit as $t \to 0^+$ (δ)

(Technical note): The treatment of the material here is informal. There is a significant amount of analysis required for a careful study. I've made some useful technical notes (insofar as they are relevant), in boxes marked '**(Technical note)**' like this one.

(Notation [Read this!]: The book uses ω for the wavenumber (the independent variable in Fourier space), whereas k is used here. The typical symbol is k; the book chooses ω, I think, because it is a 'frequency'.

Otherwise, the conventions for the Fourier transform should match the book. There may be a few small discrepancies, so be careful (if you see any, point them out).

1. Complex Fourier series

Let $f(\theta)$ be 2π-periodic. Recall that its Fourier series is

$$f(\theta) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos n\theta + b_n \sin n\theta, \quad a_n \text{ or } b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)(\cos n\theta \text{ or } \sin n\theta)\, d\theta \tag{1.1}$$

Wrting $\cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta})$ and $\sin n\theta = \frac{1}{2}e^{in\theta} - e^{-in\theta}$, we get

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\theta} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-in\theta} = \sum_{n=-\infty}^{\infty} c_n e^{-in\theta}$$

upon defining the 'complex Fourier coefficients'

$$c_n = \frac{a_n + ib_n}{2},\ c_{-n} = \frac{a_n - ib_n}{2} \text{ for } n \geq 1,\ b_0 = 0$$

By manipulating the above and (1.1), we find that

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{-in\theta}, \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{in\theta}\,d\theta. \tag{1.2}$$

This is the **complex Fourier series** (which is also defined for complex functions). Note that the complex series has + and − terms and the $\pm n$ terms both combine to give the n-th real terms. To evaluate coefficients, we can convert to a contour integral on the unit circle:

$$F(e^{i\theta}) = f(\theta).$$

This gives (With $\Gamma = \{z(\theta) = e^{i\theta},\ \theta \in [0, 2\pi]\}$)

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)(e^{i\theta})^n\,d\theta = \frac{1}{2\pi}\oint_\Gamma \frac{F(z)}{z^n}\frac{dz}{iz} \implies \boxed{c_n = \frac{1}{2\pi i}\oint_\Gamma \frac{F(z)}{z^{n+1}}\,dz.}$$

Some theory: Define the complex (L^2) inner product on $[-L, L]$ as

$$\langle f, g\rangle = \int_{-L}^{L} f\overline{g}\,dx.$$

Note the conjugate on the second argument. The basis functions

$$\phi_n(x) = e^{-inpix/L}$$

are orthogonal in this inner product:

$$0 = \langle \phi_m, \phi_n\rangle = \int_{-L}^{L} e^{-im\pi x/L}e^{in\pi x/L}\,dx = 0 \quad \text{for } m \neq n.$$

The complex Fourier series then follows directly from the fact that the ϕ_n's (for $n \in \mathbb{Z}$) are a basis for complex-valued functions $f(x)$ defined on $[-L, L]$.

Caution: Be careful with the complex inner product! It is **linear** in the first argument and **conjugate linear** in the second argument:

$$\langle cf, g\rangle = c\langle f, g\rangle, \qquad \langle f, cg\rangle = \overline{c}\langle f, g\rangle.$$

It is important to remember to **conjugate the second argument.**

2. Extending Fourier series to an infinite domain

The main idea: Fourier series and, more generally, eigenfunction bases were used to represent functions in **bounded intervals**, say $[-L, L]$:

$$f(x) = \sum_n c_n \phi_n, \quad c_n = \text{const.} \int_{-L}^{L} f(x)\phi_n(x)\, dx.$$

This suggests we can 'take the limit' as $L \to \infty$ get an infinite interval. To do so, we must contend with the fact that

- This converts the **discrete** set of eigenfunctions into a **continuous** one
- Taking the limit requires some assumptions on the function 'at $\pm\infty$'

With this, we can extend the eigenfunction theory to solve problems on infinite intervals.

The goal here: turn the Fourier series on $[-L, L]$ into the 'Fourier transform' for $(-\infty, \infty)$.

To start, consider the complex Fourier series in the interval $[-L, L]$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \qquad c_n = \frac{1}{2L} \int_{-L}^{L} f(\xi) e^{in\pi\xi/L}\, d\xi \tag{2.1}$$

Define the **wavenumber** k, as a function of n, and its change Δk in n by

$$k(n) = \frac{n\pi}{L}, \qquad \Delta k = k(n+1) - k(n) = \frac{\pi}{L}.$$

Now carefully take the limit of (2.1) as $L \to \infty$. We will need to convert a sum to an integral using a **Riemann sum** In general, for a function $g(k)$ and k depending on n,

$$\sum_{n=-\infty}^{\infty} \Delta k\, g(k(n)) \to \int_{k(-\infty)}^{k(\infty)} g(k)\, dk \text{ as } \Delta k \to 0$$

Rearranging (2.1) into this form,

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(\xi) e^{in\pi\xi/L}\, d\xi \right) e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} \left(\int_{-L}^{L} f(\xi) e^{ik(n)\xi}\, d\xi \right) e^{-ik(n)x} \\ &= \sum_{n=-\infty}^{\infty} \Delta k F_L(k), \qquad F_L(k) = \int_{-L}^{L} f(\xi) eik(n)\xi\, d\xi. \end{aligned}$$

Now take the limit of as $L \to \infty$ with $\Delta k \to 0$ and use the Riemann sum rule to get

$$f(x) \to \int_{-\infty}^{\infty} F(k) e^{-ikx}\, dk \text{ as } L \to \infty, \qquad F(k) := \lim_{L\to\infty} F_L(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{ik\xi}\, d\xi.$$

We now have a representation of f in terms of 'coefficients' $F(k)$ in a 'basis' e^{ikx}, now a continuous set (for each $k\ in \mathbb{R}$). Compare to the Fourier series:

$$F(k),\ e^{-ikx},\ k \in (-\infty, \infty) \text{ analogous to } c_n,\ \phi_n = e^{-ik(n)x},\ n \in \mathbb{Z}.$$

To extract the coefficients, we can 'take the inner product' as in the eigenfunction case. Motivated by this, we define, for $f(x)$,

$$\text{Fourier transform: } F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \mathcal{F}(f),$$

$$\text{Inverse transform: } f(x) = \int_{-\infty}^{\infty} F(k)e^{-ikx}\,dk = \mathcal{F}^{-1}(F).$$

The proof of these facts (the inverse is really the inverse, when this is defined etc.) will not be pursued here in detail; we'll see a bit of it later. Some additional notes may be added for completeness (see subsection 7.1).

3. The Fourier transform

Now the defition of the Fourier transform is motivated. They certainly deserve a box:

Definition: For a function $f(x)$ defined on $(-\infty, \infty)$, the **Fourier transform** is defined by

$$F(k) = \mathcal{F}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx. \tag{3.1}$$

The **inverse (Fourier) transform** is given by

$$f(x) = \mathcal{F}^{-1}(F) = \int_{-\infty}^{\infty} F(k)e^{-ikx}\,dk \tag{3.2}$$

For sufficiently nice input functions, the transforms (3.1) and (3.2) are well-defined and return functions. For less nice functions, they instead give **distributions** (to be addressed later).

The **inversion theorem** asserts the inverse transform is really the inverse:

$$f = \mathcal{F}^{-1}(\mathcal{F}(f)) \quad \text{for all reasonable } f.$$

Our main concern is using the Fourier transform to solve DEs that are:

- linear with constant coefficients (ODEs or PDEs!)
- On the infinite domain $(-\infty, \infty)$ with $u \to 0$ in **both limits**

It is important to note that **both properties are essential** for the Fourier transform to make things 'easy' to solve. We will find it to a be powerful but restrictive technique on its own; more work is required to get around the two listed restrictions.

Warning (notation): Unfortunately, the 'Fourier transform' has several conventions:

- The minus sign is sometimes put on $\mathcal{F}$ instead of $\mathcal{F}^{-1}$ (so $\mathcal{F}(f) = (1/2\pi)\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$)
- The factor of $1/2\pi$ is put on the inverse, or it is split into a factor of $1/\sqrt{2\pi}$ on **both halves**, or (rarely) the 2π is put in the exponential ($e^{2\pi ikx}$).

For this reason, formulas, properties etc. can differ by minus signs or factors of 2π. The differences 'cancel out when taking the transform and then inverse transforming back.

4. Properties of the transform

4.1. **Linearity:** First, note that the Fourier transform is a **linear** operator, so

$$\mathcal{F}(c_1u_1 + c_2u_2) = c_1\mathcal{F}(u_1) + c_1F(u_2).$$

This makes the Fourier transform act nicely on LCC differential equations.

4.2. **Differentiation:** Suppose u '**vanishes at** $\pm\infty$ '. That is,

$$u(x) \text{ is defined on } (-\infty,\infty) \text{ with } u \to 0 \text{ as } x \to \pm\infty. \tag{4.1}$$

Let $U(k) = \mathcal{F}(u(x))$ be the Fourier transform. To compute the transform of u', use IBP:

$$\begin{aligned}\mathcal{F}(\frac{du}{dx}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{du}{dx}e^{ikx}\,dx \\ &= \frac{1}{2\pi}u(x)e^{ikx}\Big|_{-\infty}^{\infty} - \frac{ik}{2\pi}\int_{-\infty}^{\infty} u(x)e^{ikx}\,dx \\ &= -ik\mathcal{F}(u)\end{aligned}$$

since the boundary terms vanish due to the decay assumption (4.1). Iterating this n-times:

Differentiation rule: If $u(x)$ vanishes at $\pm\infty$ then

$$\mathcal{F}(\frac{du}{dx}) = -ik\mathcal{F}(u). \tag{4.2}$$

If u and its derivatives up to order $n-1$ vanish at $\pm\infty$ then

$$\mathcal{F}(\frac{d^nu}{dx^n}) = (-ik)^n\mathcal{F}(u). \tag{4.3}$$

The Fourier transform **turns derivatives to multiplication** by $-ik$.

(Technical note:) Note $u, u', \cdots, u^{(n-1)}$ must vanish for the n-th order rule. Typically,

$$u \to 0 \text{ as } x \to \pm\infty \implies u \sim \text{constant} \implies u \text{ and all derivatives} \to 0 \text{ as } x \to \pm\infty.$$

For instance, $u = 1/x \to 0$ as $x \to \pm\infty$ and the n-th derivative decays like $1/x^{n+1}$, which goes to zero (even faster). There are pathological examples like

$$u(x) = \sin(x^2)/x \implies u'(x) = 2\cos(x^2) - \sin(x^2)/x^2 \not\to 0$$

but typically 'goes to zero' also means 'becomes flat' (all derivatives $\to 0$).

Casual derivation: The rule (4.3) can be 'derived' by differentiating the inverse transform:

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}\left(\int_{-\infty}^{\infty} U(k)e^{-ikx}\,dk\right) \\ &= \int_{-\infty}^{\infty} \frac{d}{dx}(U(k)e^{-ikx})\,dk \qquad \text{(if nice)} \\ &= \int_{-\infty}^{\infty} U(k)(-ik)e^{-ikx}\,dk \\ &= \mathcal{F}^{-1}(-ikU(k))\end{aligned}$$

and finally take $\mathcal{F}$ of both sides to get

$$\mathcal{F}(du/dx) = -ikU(k) = -ik\mathcal{F}(u).$$

The rule of thumb is that for functions with nice decay, the Fourier integral wants to be differentiated; otherwise one has to be careful. Integration by parts is safer (interchanging limits/integrals has more restrictions than IBP). You can **very easily get into trouble** by being too casual (just as we saw with differentiating Fourier series in Week 1).

4.3. **Operators.** A LCC (linear constant coefficient) differential operator like

$$L = au'' + bu' + cu$$

gets transformed into a 'multiplication' factor:

$$\begin{aligned}\mathcal{F}(Lu) &= a\mathcal{F}(u'') + b\mathcal{F}(u') + c\mathcal{F}(u)\\ &= ((-ik)^2 a + b(-ik) + c)U(k)\\ &= S(k)U(k)\end{aligned}$$

where $S(k)$ is called the (Fourier) **symbol** of the operator L. Note that the symbol is the same as the characteristic polynomial evaluated at $-ik$:

$$p(\lambda) = a\lambda^2 + b\lambda + c, \qquad S(k) = p(-ik).$$

This means that the correspondence between L and the symbol is that

$$Lu \text{ in 'physical' space} \iff S(k)\cdot U(k) \text{ in Fourier space.}$$

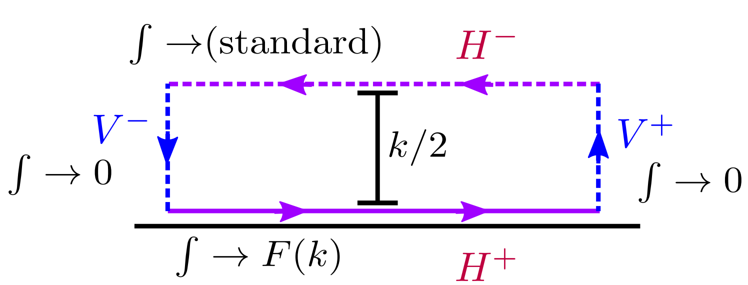
4.4. **Transform of the Gaussian:** The Gaussian function has a nice property that will be useful: the Fourier transform of a Gaussian is also a Gaussian. To prove this, let

$$f(x) = e^{-x^2}, \qquad F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}e^{ikx}\,dx.$$

We compute the transform using a contour integral:

$$\int_{-\infty}^{\infty} g(z)\,dz, \quad g(z) := e^{-z^2}e^{ikz}.$$

A box contour is necessary; let $H^\pm$ and $V^\pm$ be as in the sketch below, up to length R along the real axis and H^+ runs along $x + ib$.



To choose b, look for a value such that $g(x+ib)$ is a nice function (easier to integrate):

$$g(x+ib) = \exp(-(x+ib)^2 + ik(x+ib)) = \exp(-x^2 - 2xbi + b^2 + ikx - kb).$$

Pick $b = k/2$ to cancel out two of the terms; then

$$g(x+ib) = e^{b^2-kb}e^{-x^2}.$$

Fortunately, e^{-x^2} is easy to integrate explicitly (standard). Then

$$0 = \oint g(z)\,dz = \int_{-\infty}^{\infty} g(z)\,dz - e^{b^2-kb}\underbrace{\int_{-\infty}^{\infty} e^{-x^2}\,dx}_{\sqrt{\pi}}$$

since there are no singularities in the box (and taking $R \to \infty$ and $b = k/2$).

$$f(x) = e^{-x^2} \implies F(k) = \frac{e^{-k^2/4}}{2\sqrt{\pi}}. \tag{4.4}$$

(Intuition) sharp peaks: Consider the transform of a Gaussian of 'width' $\sqrt{2a}$:

$$f(x) = e^{-ax^2/2} \iff F(k) = \frac{1}{\sqrt{2\pi a}}e^{-x^2/(2a)}.$$

Observe that the Gaussian in Fourier space is spread out when a is small (over a 'width' of size $1/\sqrt{a}$) and vice versa. Moreover, the Fourier transform of $e^{-x^2/2}$ is itself (up to a factor) - it is 'evenly' spread out in both spaces.

In general, functions that are **sharply peaked** have 'spread out' Fourier transforms: they contain a wide range of frequencies.

4.5. **Convolution:** We will often need to take the inverse transform of a product:

$$F = \mathcal{F}(f), \quad G = \mathcal{F}(g) \implies \mathcal{F}^{-1}(F(k)G(k)) = ?$$

The important result (enough to define the result as its own object) is the following:

Convolution: The **convolution** of two functions $f(x)$ and $g(x)$ on $(-\infty, \infty)$ is

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

That is, the convolution is denoted by $f{*}g$ (f 'star g or f 'convolved with' g).

For functions that satisfy the appropriate conditions,

$$\frac{1}{2\pi}\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

That is, convolution in physical space corresponds to multiplication in Fourier space.

Proof. (Sketch, eliding technical details) The trick is to exchange the order of integration and shift the exponential to separate the transforms of f and g from the convolution:

$$\begin{aligned}\frac{1}{2\pi}\mathcal{F}(f*g) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right)e^{ikx}\,dx \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)e^{ikx}\,dx\,dy \qquad \text{(swap int. order)} \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)f(\xi)e^{ik(\xi+y)}\,d\xi\,dy \qquad \text{(shift } x\text{: } \xi = x-y)\end{aligned}$$

By shifting x, the two integration variables are now separated except in the exponential. But they can be separated there, too, leaving

$$\begin{aligned}\frac{1}{2\pi}\mathcal{F}(f*g) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)f(\xi)e^{ik\xi}e^{iky}\,d\xi\,dy \\ &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{ik\xi}\,d\xi\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)e^{iky}\,dy\right) \\ &= \mathcal{F}(f)\cdot\mathcal{F}(g)\end{aligned}$$

To be rigorous, one has to justify the exchange of integration order. □

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5. Linear, constant coefficient ODEs

5.1. ODEs. As an example, the Fourier transform is used to solve

$$\frac{d^2u}{dx^2} = e^{-x^2}, \quad u \to 0 \text{ as } x \to \pm\infty.$$

By the boundary conditions, the derivative rule applies so we may take the Fourier transform of both sides and obtain

$$(-ik)^2 U(k) = \frac{1}{2\sqrt{\pi}}e^{-k^2/4}.$$

Now divide by k^2 and take the inverse transform:

$$U(k) = -\frac{2}{\sqrt{\pi}k^2}e^{-k^2/4} \implies u(x) = -\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1}{k^2}e^{-k^2/4}e^{ikx}\,dk.$$

The point here is that the operator was converted into $-k^2$, which is easily divided to the other side; the result is then some integral expression.

Convolution: More generally, we see that for equations like

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = f(x), \quad u \to 0 \text{ as } x \to \pm\infty$$

the transformed solution is

$$U(k) = H(k)F(k), \qquad H(k) := \frac{1}{a(-ik)^2 + b(-ik) + c}, \quad F = \mathcal{F}(f)$$

where $H(k) = 1/S(k)$ is the reciprocal of the symbol. Now use the convolution rule to get

$$u(x) = \mathcal{F}^{-1}(H(k)F(k)) = \frac{1}{2\pi}h*f, \qquad h = \mathcal{F}^{-1}(H).$$

The function h is easily computed using a contour integral since it has the form

$$h = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{q(x)}\, dx.$$

Notice that $\frac{1}{2\pi}h(x-y)$ is a Green's function for the problem:

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x-y)f(y)\, dy.$$

In particular, this suggests that $h(x-y)$ is the solution to the problem

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = \delta(x-y), \quad u \to 0 \text{ as } x \to \pm\infty$$

i.e. the response of the system to a source of unit mass, concentrated at the point $x = y$. We'll address the issue in more detail shortly (the δ suggests proceeding with care!).

5.2. **Projection.** It is worth comparing to the bounded case. Consider

$$Lu = f, \qquad \text{(hom. BCs at } a \text{ and } b\text{)}$$

(with L self-adjoint for simplicity) with eigenfunctions ϕ_n $(n \geq 1)$. The solution is

$$u = \sum_{n\geq 1} c_n\phi_n(x)$$

where the c_n's are obtained by **projecting onto the n-th eigenfunction**:

$$\langle Lu, \phi_n\rangle = \langle f, \phi_n\rangle \implies \text{ equations for } c_n,\ n = 1, 2, \cdots.$$

For the infinite domain, the procedure is the same, but for a continuous set of 'eigenfunctions'. Define the inner product and functions

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx, \qquad \phi_k = e^{-ikx},\ k \in \mathbb{R}.$$

Then the LCC equation

$$Lu = f, \qquad u \to 0 \text{ as } x \to \pm\infty$$

has a solution

$$u(x) = \int_{-\infty}^{\infty} U(k)e^{-ikx}\, dx$$

and the $U(k)$'s are found by 'projecting' onto ϕ_k, for real k. In terms of the Fourier transform,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Lu(x)e^{ikx}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ikx}\, dx$$
$$\implies S(k)U(k) = F(k)$$

which, in terms of the projection, is a process for finding 'coefficients' $U(k)$:

$$\frac{1}{2\pi}\langle Lu, \phi_k\rangle = \frac{1}{2\pi}\langle f, \phi_k\rangle \implies \text{ equations for } U(k),\ k \in \mathbb{R}.$$

That is, we are still taking the projection as with eigenfunctions, except now over a continuous set (yielding functions of k instead of sums over k).

6. The heat equation

We get a more significant result by solving the heat equation:

$$u_t = u_{xx}, \quad x \in (-\infty, \infty),\ t > 0$$
$$u \to 0 \text{ as } x \to \pm\infty$$
$$u(x,0) = f(x)$$

noting that the domain is now the entire (real) line. Take the Fourier transform in x and apply the derivative rule (again, justified because of the BCs!) to get

$$\mathcal{F}(u_t) = \mathcal{F}(u_{xx})$$
$$\implies \frac{1}{2\pi}\int_{-\infty}^{\infty} u_t(x,t)e^{ikx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_{xx}e^{ikx}\,dx$$
$$\implies \frac{dU}{dt} = (-ik)^2 U = -k^2 U. \tag{6.1}$$

where $U = U(k,t)$. Just as projecting with $\langle \cdot, \phi_n \rangle$ produced ODEs for coefficients $c_n(t)$, the Fourier transform produces ODEs for $U(k,t)$ in time for each wavenumber k.
Now Fourier transform the IC to get initial data for the ODE:

$$u(x,0) = f(x) \implies U(k,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = F(k).$$

Note that this shows that in Fourier space, the solution at each wavenumber **evolves independently** - the Fourier transform is exactly what is needed to disentangle the PDE into independent ODEs. The solution $U(k,t)$ depends only on the value of the IC at k (just $F(k)$).

To complete the solution, solve the ODE (6.1) with initial condition $U(k,0) = F(k)$:

$$U(k,t) = F(k)e^{-k^2 t}. \tag{6.2}$$

The inverse transform of $e^{-k^2 t}$ is found by rescaling the Gaussian formula (left as an exercise):

$$\mathcal{F}^{-1}(e^{-k^2 t}) = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-x^2/4t}$$

Now apply the convolution rule to inverse transform (6.2):

$$u(x,t) = \frac{1}{2\pi} f * \mathcal{F}^{-1}(e^{-k^2 t})$$
$$= \int_{-\infty}^{\infty} f(y)\left(\frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}\right)\,dy.$$

Observe that we have now derived a sort of Green's function for the solution:

$$u(x,t) = \int_{-\infty}^{\infty} f(y)G(x-y,t)\,dy, \qquad G(x-y) := \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}. \tag{6.3}$$

The function G is called the **fundamental solution** (the book calls this the 'influence function') for the heat equation in $\mathbb{R}$:

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}. \tag{6.4}$$

It is the Green's function for the IBVP where the input is the initial condition (no source term; the Green's function with a source is different (see 6.1). From formally plugging in δ's we find that $G(x-x_0,t)$ solves the equation

$$\begin{aligned} u_t &= u_{xx}, \quad x \in (-\infty,\infty),\, t>0 \\ u &\to 0 \text{ as } x \to \pm\infty \\ u(x,0) &= \delta(x-x_0) \end{aligned}$$

That is, $G(x-x_0,t)$ is the solution to the heat equation in an infinite interval, where the IC has unit mass, concentrated at $x=x_0$ - for instance, if a drop of dye is injected into a(n infinite) container and allowed to diffuse.

(Technical note): In fact, one can show rigorously that

$$\lim_{t\searrow 0} G(x,t) = \delta(x).$$

As t decreases to zero, the exponential becomes more and more sharp: the width scales like $\sqrt{t}\to 0$ and the height scales like $1/\sqrt{t}\to\infty$, and

$$\int_{-\infty}^{\infty} G(x,t)\,dx = 1 \text{ for all } t>0.$$

With these ingredients, we can make rigorous sense of the δ limit as $t\to 0$.

6.1. **A note on source terms.** A source term can be added without much trouble, e.g.

$$\begin{aligned} u_t &= u_{xx} + h(x,t), \quad x\in(-\infty,\infty),\, t>0 \\ u &\to 0 \text{ as } x\to\pm\infty \\ u(x,0) &= f(x) \end{aligned}$$

Transform and solve (with an integrating factor) to get

$$\begin{aligned} U_t &= -k^2 U + H(k,t), \qquad H = \mathcal{F}(h), \\ &\implies (e^{k^2 t}U)_t = e^{k^2 t} H(k,t) \end{aligned}$$

which can be solved and inverse transformed. The solution becomes interesting if

$$h = h(x)\delta(t-t_0).$$

i.e. it is a source added instantly at $t=t_0$. Then

$$\begin{aligned} U(k,t) &= F(k)e^{-k^2 t} + e^{-k^2 t}\int_0^t e^{k^2 s} H(k)\delta(s-t_0)\,ds \\ &= F(k)e^{-k^2 t} + H(k)e^{-k^2(t-t_0)}. \end{aligned}$$

Now inverse transform to get

$$u(x,t) = \underbrace{\int_{-\infty}^{\infty} f(y)G(x-y,t)\,dy}_{\text{IC part}} + \underbrace{\int_{-\infty}^{\infty} h(y)G(x-y,t-t_0)\,dy}_{\text{source part}}. \tag{6.5}$$

This suggests a Green's function can be constructed for sources as well, which is indeed the case. For details, see Chapter 11 of Haberman or look up **Duhamel's principle**.

7. Misc. notes

7.1. **Orthogonality for the Fourier transform.** You may note that, putting the transform/inverse together we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{ikx}e^{-ik\xi}\,dk\,d\xi = \int_{-\infty}^{\infty} f(\xi)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)}\,dk\right)\,d\xi$$

at least informally, by rearranging the integral. This says (roughly) that

$$\int_{-\infty}^{\infty} e^{ikx}e^{-ik\xi}\,dk = \delta(x-\xi) \tag{7.1}$$

which is a continuous analogue of the discrete orthogonality property

$$\int_{-L}^{L} e^{im\pi x/L}e^{-in\pi x/L}\,dx = 0 \text{ if } m \neq n.$$

To make precise sense of this and to prove the inversion formula, one has to be more careful and 'smooth out' the integral a bit to avoid the δ and allow the integral to converge (since (7.1) is questionable). See the **Fourier inversion theorem** for details.