MATH 5410 LECTURE NOTES
FOURIER-LIKE TRANSFORMS

Topics covered

- Equations in $[0, \infty)$
  - Sine and cosine transforms
  - Examples (Dirichlet BC at $x = 0$)
  - Symmetry tricks with Green's functions
  - Extending the idea to other problems (Hankel, etc.)
- A few extra examples
  - Standard Fourier transform example with contour integral
  - Bonus example: odd order derivatives

1. Continuous separation of variables

We saw that the Fourier transform is analogous to a continuous version of the eigenfunction method with continuous basis $\phi_k = e^{-ikx}$. The same approach works for separation of variables to find the right continuous basis for other domains and boundary conditions.

1.1. A first example.

Consider the heat equation in a'half-infinite' domain $[0, \infty)$:

$$\begin{aligned} u_t &= u_{xx}, \quad x \in [0, \infty),\ t > 0 \\ u(0,t) &= 0, \quad u(x,t) \to 0 \text{ as } x \to \infty \\ u(x,0) &= f(x) \end{aligned}$$

Note that there is a **Dirichlet BC** (homogeneous) at $x = 0$. Proceed by using SoV:

$$u = g(t)\phi(x) \implies \frac{g'(t)}{g(t)} = \frac{\phi''(x)}{\phi(x)}.$$

Only the BC at $x = 0$ can be imposed directly; the eigenvalue problem is then

$$-\phi'' = \lambda\phi, \quad \phi(0) = 0, \quad (\phi \text{ bounded}).$$

This yields eigenfunctions

$$\phi_k = \sin kx, \quad k > 0.$$

From here one could proceed by continuing with SoV to get separated solutions

$$u = e^{-k^2 t} \sin kx$$

and then using 'superposition' to construct the full solution by integrating over k:

$$u(x,t) = \int_0^\infty F(k) e^{-k^2 t} \sin kx \, dk, \qquad f(x) = \int_0^\infty F(k) \sin kx \, dk.$$

Similarly, for a **Neumann** boundary condition

$$u_x(0,t) = 0,$$

the eigenvalue problem and solution are

$$-\phi'' = \lambda\phi \ \ \phi'(0) = 0, \ (\phi \text{ bounded}) \implies \phi_k = \cos kx.$$

1.2. **The sine/cosine transforms.** Now we use this idea to extend the eigenfunction method (from bounded domains) to continuous sets of eigenfunctions. Recall that we found eigenfunctions ϕ_n and took inner products $\langle \cdot, \phi_n \rangle$ of the PDE.

Motivated by this, we define $\langle \cdot, \phi_k \rangle$ (with a constant) to be a 'new' transform:

Sine transform: Let f be defined on $[0, \infty)$. The **sine transform** of f and its inverse are

$$F(k) = S[f] = \frac{2}{\pi} \int_0^\infty f(x) \sin kx \, dx, \tag{1.1}$$

$$f(x) = \int_0^\infty F(k) \sin kx \, dk.$$

(the notation is less standard her, unlike the more important Fourier transform).
The **cosine transform**, similarly, and inverse are

$$F(k) = C[f] = \frac{2}{\pi} \int_0^\infty f(x) \cos kx \, dx, \tag{1.2}$$

$$f(x) = \int_0^\infty F(k) \cos kx \, dk.$$

• The sine transform is used for DEs in $[0, \infty)$ with a **Dirichlet BC** at $x = 0$.

• The cosine transform is used for DEs in $[0, \infty)$ with a **Neumann BC** at $x = 0$.
• $S[f]$ and $C[f]$ are (up to a constant) the **Fourier transform** of the **odd extension** and **even extension** of f, respectively.

For the last claim, let f_o be the odd extension. Then

$$\begin{aligned}
\mathcal{F}(f_o) &= \frac{1}{2\pi} \left(\int_{-\infty}^0 f(x) e^{ikx} \, dx + \int_0^\infty f(x) e^{ikx} \, dx \right) \\
&= \frac{1}{2\pi} \left(\int_0^\infty (-f(x)) e^{-ikx} \, dx + \int_0^\infty f(x) e^{ikx} \, dx \right) \\
&= \frac{1}{\pi} \int_0^\infty f(x) \sin kx \, dx.
\end{aligned}$$

The extra factor of 2 in (1.1) is to make the inverse transform have no coefficient. A similar calculation for the even extension ($f_e(-x) = f_e(x)$) shows that

$$\mathcal{F}(f_e) = \frac{1}{2\pi} \int_{-\infty}^\infty f_e(x) e^{ikx} \, dx = \frac{1}{\pi} \int_0^\infty f(x) \cos kx \, dx.$$

To use the transform, we need rules for the derivative to compute $\langle u_{xx}, \phi_k \rangle$. This is derived using integration by parts. If u vanishes at ∞ then

$$S[u_x] = \frac{2}{\pi} \int_0^\infty u_x \sin kx \, dx = \frac{2}{\pi} u \sin kx \Big|_{x=0}^\infty - \frac{2k}{\pi} \int_0^\infty u \cos kx \, dx = -kC[u]$$

Note that **the cosine transform appears**, which is not good (mixing the two transforms). However, applying the rule again (details: exercise) gives

$$S[u_{xx}] = \frac{2k}{\pi} u\Big|_{x=0} - k^2 S[u]. \tag{1.3}$$

Note that the BC at $x = 0$ must be **homogeneous** for the boundary terms to vanish. Since u_{xx} transforms into a multiple of the transform of u, we can use the sine transform when **only even derivatives are involved**.

Example (heat equation): The sine transform is used to solve

$$\begin{aligned} u_t &= u_{xx}, \quad x \in [0, \infty),\ t > 0 \\ u(0,t) &= 0 \\ u(x,t) &\to 0 \text{ as } x \to \infty \\ u(x,0) &= f(x) \end{aligned} \tag{1.4}$$

Take the sine transform of the PDE/ICs (equivalently, $\frac{2}{\pi}\langle \cdot, \phi_k \rangle$ with $\phi_k = \sin kx$) to get

$$U_t = \frac{2k}{\pi} u(0,t) - k^2 U, \quad U(k,0) = F(k)$$

But $u(0,t) = 0$ so $U_t = -k^2 U$; this can be solved to obtain the transformed solution

$$U(k,t) = F(k) e^{-k^2 t}, \qquad k > 0.$$

At this point, we could use the convolution rule for the sine transform (see Haberman). Here we use a symmetry trick to make use of the Fourier transform. Let $F_o(k)$ be the odd extension of $F(k)$. Then the odd extension of U is

$$U_o(k,t) = F_o(k) e^{-k^2 t}, \qquad k \in \mathbb{R} \tag{1.5}$$

since $e^{-k^2 t}$ is even. Note that by the box of the previous page,

$$\mathcal{F}(u_o(x,t)) = U_o(k,t) \text{ where } u_o = \text{odd ext. of } u.$$

Now use the convolution rule for $\mathcal{F}$ on (1.5) to invert (as earlier for the heat eq. on $(-\infty, \infty)$):

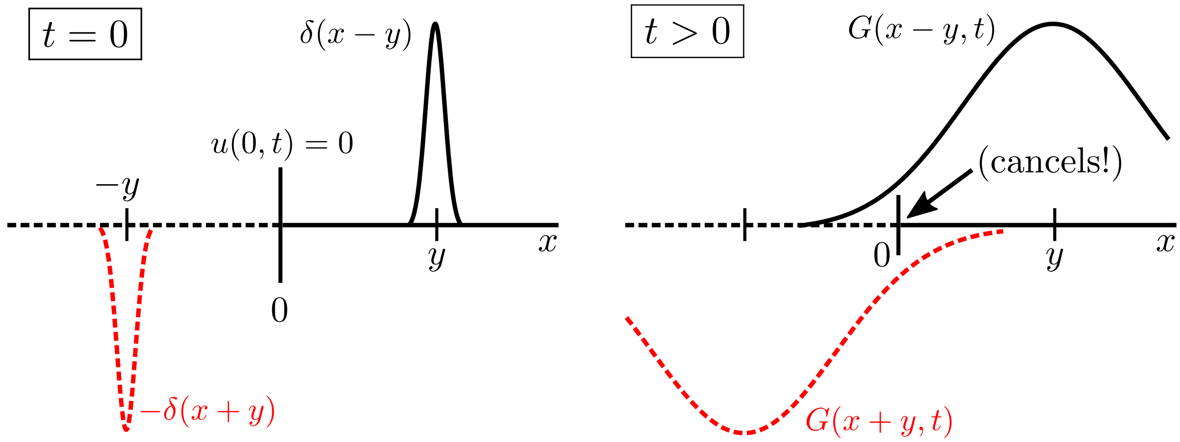
$$u_o(x,t) = f_o * G(x,t) = \int_{-\infty}^{\infty} f_o(y) G(x-y,t)\, dy, \qquad G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Now we must translate back to $[0, \infty)$ from the extension. Since f_o is odd, the integral can be written over $[0, \infty)$ by changing variables in the $(-\infty, 0]$ half:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{0} f_o(y) G(x-y,t)\, dy + \int_0^{\infty} f_o(y) G(x-y,t)\, dy \\ &= \int_0^{\infty} f(y) \left[G(x-y,t) - G(x+y,t) \right] dy \\ &= \int_0^{\infty} f(y) K(x,y,t)\, dy. \end{aligned}$$

The function K is the Green's function for input IC $f(x)$ for this problem.

Remark (symmetry tricks): This solution illustrates a useful tool: use **symmetry** to superimpose Green's functions to build new ones with desired BCs. What we are doing is placing an IC of a δ for the heat equation on $(-\infty, \infty)$ at point y and $-\delta$ at point $-y$ (see below). The two contributions cancel by symmetry at $x = 0$, forcing u to be zero there. Then the $x > 0$ half is the solution we want for the $+y$ IC!



1.3. **Sine transform: inhomogeneous BC.** We can also solve the problem with an inhomogeneous BC using the inner product: boundary terms at $x = 0$ **do not vanish**. Consider

$$\begin{aligned} u_t &= u_{xx} + h(x,t), \quad x \in [0,\infty),\ t > 0 \\ u(0,t) &= g(t) \\ u(x,t) &\to 0 \text{ as } x \to \infty \\ u(x,0) &= f(x) \end{aligned} \tag{1.6}$$

Take the sine transform of both sides of the PDE/ICs; note that the IBP on the $u_x x$ term (the derivative rule) gives a boundary term (see (1.3)):

$$\frac{2}{\pi}\int_0^\infty u_t \sin kx\, dx = \frac{2}{\pi}\int_0^\infty u_{xx} \sin kx\, dx$$

$$\implies U_t = \frac{2k}{\pi} g(t) - k^2 U + H(k,t), \qquad U(k,0) = F(k).$$

where $U = S[u]$ and $H = S[h]$. This ODE is first order, linear, so it can be solved, and then U can be inverse transformed to get the solution.

Remark (computing transforms): To compute transforms and inverse transforms, one typically has to evaluate with a contour integral, like

$$\int_0^\infty (\cdots)\, dx = \int_\Gamma (\cdots)\, dz, \quad \Gamma = \text{ real axis, } 0 \to \infty$$

and so on. Using the rules for the Fourier transform (e.g. in the HW) and some standard transforms, the work can be simplified. Tables of transforms save work. For instance,

$$u_t = u_{xx} + 3e^{-7x-t}, \quad x \in [0,\infty),\ \cdots$$

$$\implies U_t = \frac{2k}{\pi} u(0,t) - k^2 U + \frac{6}{\pi} e^{-t} \frac{k}{49 + k^2}$$

using the formula $S[e^{-ax}] = \frac{2}{\pi} x/(a^2 + x^2)$.

1.4. **Cosine transform; limitations.** The sine transform and cosine transform work when

i) The interval is $[0,\infty)$ (half-infinite)

ii) Dirichlet BCs ($u(0,t)$) for the sine transform or Neumann BCs ($u_x(0,t)$) for cosine

ii) All derivatives are even or all derivatives are odd

If (ii) fails, we are left with unknown terms, e.g.

$$S[u_{xx}] = \frac{2k}{\pi}u(0,t) - k^2 U$$

requires $u(0,t)$ to be specified. If (iii) fails, then the transforms get mixed, e.g.

$$u_t = u_{xx} + u_x \implies (S[u])_t = -k^2 S[u] - kC[u] + \cdots .$$

which gives an equation for two functions $S[u]$ and $C[u]$. Other tools must be used instead.

1.5. **Other transforms.** Note that in both cases, separation of variables suggests the appropriate 'kernel' for the integral transform, e.g. for the sine transform,

$$\phi'' = -\lambda\phi, \quad \phi \text{ bded.}, \ \phi(0) = 0 \implies \phi_k = \sin kx.$$

What about for other Sturm-Liouville problems? Take, for example, the heat equation with radial symmetry in 2d in the whole plane:

$$u_t = \frac{1}{r}(ru_r)_r, \qquad r \in [0,\infty).$$

Separation of variables and a guess at BCs gives the eigenvalue problem

$$\phi'' + \frac{1}{r}\phi' + \lambda\phi = 0, \quad \phi \text{ finite for all } r.$$

Imposing the implied constraint to stop $|\phi(0)| = \infty$, we get (with J_0 the usual Bessel function)

$$\phi_k(r) = J_0(kr).$$

The relevant inner product (weighted, $[0\infty)$ with $\sigma = r$ or L^2 in polar coords.) is

$$\langle f, g\rangle = \int_0^\infty f(r)g(r)r\,dr$$

which leads to the **Hankel transform** $\langle\cdot,\phi_k\rangle$:

$$F(k) = C\int_0^\infty f(r)J_0(kr)r\,dr.$$

Of course, one then has to derive the relevant properties of the inverse transform to compute solutions (not pursued here). The point is that continuous sets of eigenfunctions/values can be used just as discrete sets were in bounded domains - one just has to do more work to deal with the integral and its inverse transform.

Remark: In fact, the Hankel transform is just the 2d Fourier transform (i.e. in $\mathbb{R}^2$ instead of $\mathbb{R}$) for radially symmetric functions. Thus, it inherits the nice properties of the Fourier transform (e.g. the inverse transform also looks like $f(r) = C\int_0^\infty F(k)J_0(kr)\,dr$).

2. Misc examples for Fourier transforms

2.1. A typical ODE (use of contour integrals).

$$u'' + 4u' + u = f(x), \quad x \in (\infty, \infty),\ t > 0$$
$$u \to 0 \text{ as } x \to \pm\infty$$

We find the Green's function. Take the Fourier transform to get

$$(-k^2 - 4ik + 1)U(k) = F(k) \implies U(k) = G(k)F(k), \quad G(k) := -\frac{1}{k^2 + 4ik - 1}.$$

The solution is then

$$u(x) = \frac{1}{2\pi} g * f = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x-y)f(y)\,dy, \qquad g = \mathcal{F}^{-1}(G).$$

To get the Green's function, we must evaluate the contour integral

$$I = -\int_{-\infty}^{\infty} \frac{e^{-ixz}}{z^2 + 4iz - 1}\,dz.$$

The denominator is a polynomial, so try a semicircle. The sign of x determines the semi-circle needed. Recall that

$$|e^{-ixz}| = e^{-xR\sin\theta} \leq 1 \text{ for } \begin{cases} \theta \in [0,\pi] & x \leq 0 \\ \theta \in [\pi, 2\pi] & x \geq 0 \end{cases}$$

Check that in either case, for the correct semi-circle C_R,

$$\Big|\int_{C_R} \cdots\, dz\Big| \leq \frac{\pi R}{R^2 - 4R - 1} \sim \frac{\pi}{R} \text{ as } R \to \infty$$

For $x \leq 0$, pick C_R to be the upper half semicircle; then

$$2\pi i \sum \text{res.} = \oint \cdots\, dz \to 0 + I \text{ as } R \to \infty.$$

The poles are at $z = i(-2 \pm \sqrt{3})$. None are inside so

$$x \geq 0 \implies I = 0.$$

For $x \leq 0$ pick C_R to be the lower half semicircle. Taking the closed contour counter-clockwise,

$$2\pi i \sum \text{res.} = \oint \cdots\, dz \to 0 - I \text{ as } R \to \infty.$$

Both poles $z_1 = i(-2 + \sqrt{3})$ and $z_2 = i(-2 - \sqrt{3})$ are inside so

$$x \leq 0 \implies I = 2\pi i \left(\frac{e^{-ixz}}{2z + 4i}\Big|_{z_1} + \frac{e^{-ixz}}{2z+4i}\Big|_{z_2} \right) = \frac{\pi}{\sqrt{3}}\left(e^{x(-2+\sqrt{3})} - e^{x(-2-\sqrt{3})}\right)$$

Thus the solution is

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{x} g_+(x-y)f(y)\,dy, \qquad g_+(x) = \frac{\pi}{\sqrt{3}}\left(e^{x(-2+\sqrt{3})} - e^{x(-2-\sqrt{3})}\right).$$

since $g(x) < 0$ for $x < 0$ (the + has been included for emphasis).

Remark: The various rules for the Fourier transform could also be used (but are derived by calculations like above). By transforming $ik \to s$, this also becomes an 'easy' Laplace transform to look up or compute directly (to be detailed...).

2.2. **Bonus KdV example (what about odd order derivatives?:)** Some transforms lead to difficult to invert solutions - even ones that do not converge in the normal sense. The 'linearized KdV (Kortweig deVries) equation is

$$\begin{aligned} u_t = u_{xxx}, \quad x \in (\infty, \infty), \\ u \to 0 \text{ as } x \to \pm\infty \\ u(x,0) = f(x) \end{aligned}$$

Take the Fourier transform:

$$U_t = -ik^3 U, \quad U(k,0) = F(k)$$
$$U(k,t) = F(k)e^{-ik^3 t} = F(k)G(k), t, \qquad G(k,t) := e^{-ik^3 t}.$$

Thus the solution is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x-y,t)f(y)\,dy, \quad g(x,t) = \mathcal{F}^{-1}(G).$$

The inverse transform is not so nice to compute:

$$g(x,t) = \int_{-\infty}^{\infty} e^{-ik^3 t}e^{-ikx}\,dk = \lim_{L\to\infty}\int_{-L}^{L} e^{-ik^3 t}e^{-ikx}\,dk.$$

It does not converge, so one has to go back to the symmetric limit $[-L, L] \to (-\infty, \infty)$. These sorts of non-convergent integrals show up often; one has to be careful to use them correctly (follow the transform rules).

There is an indirect way to simplify here: instead of computing $\mathcal{F}^{-1}$, find an **ODE** for the inverse transform. To start, consider the ODE (the **Airy equation**)

$$y'' - xy = 0, \qquad y \to 0 \text{ as } x \to \pm\infty. \tag{2.1}$$

In turns out this equation has a unique solution $\mathrm{Ai}(x)$ (the **Airy function**). Now use the 'derivative rule for the inverse',

$$\mathcal{F}(xf) = -i\frac{dF}{dk}$$

and take the Fourier transform of the Airy equation (2.1) to get

$$-k^2 Y + i\frac{dY}{dk} = 0 \implies Y(k) = Ce^{-ik^3} \implies y(x) = C\int_{-\infty}^{\infty} e^{-ik^3}e^{ikx}\,dx.$$

It follows that the Green's function for the linearized KdV equation can be written in terms of the Airy function. This settles the 'does not converge' issue of the inverse transform by indirectly finding the inverse as the solution to an ODE that can be shown to have a solution. Deriving properties of $\mathrm{Ai}(x)$, requires some asymptotics for (2.1)).

Remark: The calculation above also illustrates the principle that while even order derivatives with a positive coefficient ($au_{xx} \to -ak^2 U$ etc.) lead to decaying terms in the Fourier transform, odd order derivatives lead to **oscillating but not decaying terms**.

For this reason, even order derivatives with + coefficients (diffusion) are nice, while other terms are not so nice (which makes sense since diffusion wants to spread out solutions!).