MATH 5410 LECTURE NOTES
THE LAPLACE TRANSFORM

Topics covered

- Laplace transform (idea)
- Fundamentals
  - Definition, inverse transform
  - Properties (rules for transforms)
- Solving LCC IVPs
  - The approach
  - Application: resonance and poles
- Solving PDEs
  - The heat equation on a half-infinite interval
  - How is this different from Fourier? (BCs vs. ICs)
  - Transport equation

1. Laplace transform

1.1. **Introduction.** The Fourier and related transforms can be used for **boundary value problems** on an infinite interval. An **initial value problem** is different, such as

$$u'(t) = f(t), \quad u(0) = a, \quad t \in [0, \infty).$$

The interval is the same as for the sine/cosine transforms ($[0,\infty)$), but boundary conditions are imposed only at $t = 0$. The solution is determined by its initial values: we **cannot also impose BCs at** ∞. The 'eigenvalue problem' for $u' = f(t)$ is

$$\frac{d\phi}{dt} = \lambda\phi, \ t \in [0,\infty), \ \phi \text{ bounded} \quad \Longrightarrow \quad \phi = e^{-st}, \ s > 0.$$

We might hope the right continuous basis is $\phi_s = e^{-st}$, which is indeed the case. Due to the IVP structure, the transform works a bit differently; it is not quite analogous to the eigenfunction approach.

Laplace transform: Let $f(t)$ be defined on $[0,\infty)$. The **Laplace transform** of f is

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt \tag{L}$$

and the inverse transform is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds. \tag{IL}$$

where the contour is a line in the $+i$ direction with c chosen so that the line is to the right of all singularities of $F(s)$.

The <0 intuition: The Laplace transform is 'one-sided'.[1] It sees only the function $f(t)$ in the range $t>0$. You can think of $\mathcal{L}$ as acting on functions $f(t)$ 'set to zero' for $t<0$, e.g.

$$\mathcal{L}[e^t] = \text{transform of } \begin{cases} 0 & t<0 \\ e^t & t>0 \end{cases}.$$

Derivation: The inverse transform requires some explanation due to the contour (later). The formulas can be derived from the Fourier transform with the substitution

$$x \to t, \quad ik \to s \tag{1.1}$$

Important (nice functions): The Laplace transform is defined for all 'nice' enough functions at least far enough 'to the right' in the complex plane, i.e. for

$$F(s) \text{ defined for } \text{Re}(s) > a.$$

The e^{-st} can help the integral (L) converge. A function f is 'nice' if it is **continuous except at a set of jumps** and grows at most exponentially, so a large enough s undoes its growth:

$$\lim_{t\to\infty} \frac{f(t)}{e^{at}} = 0 \implies F(s) \text{ is defined for } \text{Re}(s) > a.$$

Practical note: For most problems, $\mathcal{L}$ and $\mathcal{L}^{-1}$ can be 'blindly' computed by consulting a table or transferring results from the Fourier transform (see any reference for a table). The contour integral (IL) is important, however, for solving PDEs and can provide significant insight into solutions.

1.2. **General properties.** The basic properties are derived in the same way as the Fourier transform (either manipulating the formula directly or using contour integration). Some fundamental properties of the transform:

- **Linearity:** The Laplace transform is a linear operator.

- **Derivatives:** Like the sine transform, integration by parts leaves some boundary terms at $t=0$ behind. The derivative rule (see proof below) is

$$\mathcal{L}\Big[\frac{df}{dt}\Big] = -f(0) + s\mathcal{L}[f(t)] \tag{1.2}$$

so t-derivatives correspond to multiplication by s **(plus boundary terms)**. Iterating,

$$\mathcal{L}\Big[\frac{d^n f}{dt^n}\Big] = -f^{(n-1)}(0) - sf^{(n-2)}(0) - \cdots - s^{n-1}f(0) + s^n\mathcal{L}[f(t)] \tag{1.3}$$

[1] There are variants, such as the 'two-sided' Laplace transform that integrates over $(-\infty,\infty)$. The term 'Laplace transform' is reserved for the one-sided transform.

Proof. The proof is straightforward and worth knowing. Assume that f is 'nice' and continuous (for simplicity). The integral has to be checked to converge, so take the limit carefully:

$$\begin{aligned}\mathcal{L}[f'(t)] &= \int_0^\infty e^{-st} f'(t)\, dt \\ &= \lim_{b\to\infty} e^{-bt} f(t) - f(0) - \lim_{b\to\infty} \int_0^b (-se^{-st}) f(t)\, dt \\ &= -f(0) + s \lim_{b\to\infty} \int_0^b e^{-st} f(t)\, dt \\ &= -f(0) + s\mathcal{L}[f(t)].\end{aligned}$$

The first limit is zero by the bound on f if $s > a$ since

$$s > a \implies |e^{-st} f(t)| \le Ce^{-(s-a)t} \to 0 \text{ as } t \to \infty.$$

This bound also verifies that the integral converges ($e^{-st} f(t)$ decays exponentially). □

• **Decay and nice inversion:** A useful theorem is that

$$\text{if } f \text{ is nice and } F(s) = \mathcal{L}(f) \text{ then } \lim_{s\to\infty} F(s) = 0. \tag{1.4}$$

Equivalently, the version in the other direction tells us **when functions of s are not transforms of nice functions**:

$$\lim_{s\to\infty} F(s) \neq 0 \implies \mathcal{L}^{-1}(F) \text{ is not nice, or not a function, or...}$$

For instance,

$$\lim_{s\to\infty} \frac{1}{s-a} = 0 \implies \mathcal{L}^{-1}(\frac{1}{s-a}) \text{ is nice } (= e^{at}).$$

but $F(s) = 1$ does not decay, so it can't be the transform of a nice function:

$$\lim_{s\to\infty} 1 \neq 0 \implies \mathcal{L}^{-1}(1) = \text{not nice}.$$

We'll see that $\mathcal{L}^{-1}(1) = \delta(t)$, so the inverse transform is a distribution (not a function).

1.3. **Transform rule:** The Laplace transform has a number of nice standard transforms, very similar to the Fourier transform. A few are listed below (proofs left as exercises). The basic transform rule is

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}.$$

Thus, exponentials in the t-space correspond to **simple poles** in s-space. One can show that there is a dual to the derivative rule,

$$(-t)f(t) = \mathcal{L}^{-1}(\frac{dF}{ds})$$

from which it follows that poles of order $n+1$ correspond to $t^n e^{at}$:

$$\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}.$$

The formulas also apply for complex exponentials, so

$$\mathcal{L}[e^{(a+bi)t}] = \frac{1}{s-(a+bi)} = \frac{s-a+bi}{(s-a)^2+b^2}.$$

In particular, we can take real and imaginary parts to get

$$\mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}, \quad \mathcal{L}[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}.$$

• **Shift rule:** The Laplace transform obeys the shift rule

$$\mathcal{L}(f(t-t_0)H(t-t_0)) = e^{-st_0}F(s) \qquad \text{for } t_0 > 0.$$

where H is the Heaviside function (the function f has to 'switch on' at t_0). Unlike the Fourier transform, the shift rule 'cuts off the function' before t_0. You can think of this as the translation of the 'zero for $t < 0$' extension:

$$\begin{cases} f(t) & t > 0 \\ 0 & t < 0 \end{cases} \xrightarrow{\text{(translate by } t_0)} \begin{cases} f(t-t_0) & t > t_0 \\ 0 & t < t_0 \end{cases} = f(t-t_0)H(t-t_0).$$

An example of a typical use (note that $\mathcal{L}^{-1}(1/(s-3)^2) = te^{3t}$):

$$\mathcal{L}^{-1}(\frac{e^{-2s}}{(s-3)^2}) = \mathcal{L}^{-1}(e^{-2s}F(s)) = (t-2)e^{3(t-2)}H(t-2)$$

Any piecewise defined function can be written in terms of Heaviside functions, e.g.

$$f(t) = \begin{cases} \sin t & t < 3 \\ t^2 & t > 3 \end{cases} \implies f(t) = \sin t + (t^2 - \sin t)H(t-3).$$

• **Dirac delta:** For the delta, use the sifting property to (informally) compute

$$\mathcal{L}(\delta(t-t_0)) = \int_0^\infty \delta(t-t_0)e^{-st}\,dt = e^{-st_0}.$$

In particular, just like the Fourier transform,

$$\mathcal{L}(\delta(t)) = 1.$$

That is, the Laplace transform of δ is a constant function. This means that LCC ODEs (and other DEs) are easier to work in s-space, since **distributions** become (nice) **functions**.

1.4. **Convolutions:** For functions $f(t), g(t)$ defined on $[0,\infty)$ define the convolution

$$f * g = \int_0^t f(t_0)g(t-t_0)\,dt_0 = \int_0^t f(t-t_0)g(t_0)\,dt_0. \tag{1.5}$$

It is not hard to show this is equivalent to the full convolution of the 'zero for $t < 0$' extensions, so it is not really a new definition:

$$f * g = \int_{-\infty}^{\infty} \overline{f}(t_0)\overline{g}(t-t_0)\,dt_0 \text{ where } \overline{f} = \begin{cases} 0 & t<0 \\ f(t) & t>0 \end{cases}, \cdots .$$

With this definition, the convolution rule is (nearly) the same as the Fourier transform:

$$\mathcal{L}[f*g] = \mathcal{L}[f]\mathcal{L}[g] \tag{1.6}$$

This gives a way of inverse transforming general products (same idea as the Fourier transform). For example, if $F(s) = \mathcal{L}(f(t))$ then

$$\mathcal{L}^{-1}(\frac{F(s)}{s-1}) = e^t * f(t) = \int_0^t e^{t-t_0} f(t_0)\,dt_0.$$

Warning: The convolution 'definition' depends on the domain of the functions. Since f and g are functions defined for $t > 0$ here, we use (1.5). This is the right meaning of $*$ for the Laplace transform.
For the Fourier transform, use $\int_{-\infty}^{\infty}$ (since f, g are defined on all of $\mathbb{R}$).

Proof. Similar to the Fourier transform, but we must be more careful with integration limits.

$$\begin{aligned} \mathcal{L}[f*g] &= \int_0^\infty \left(\int_0^t f(y)g(t-y)\,dy \right) e^{-st}\,dt \\ &= \int_0^\infty \int_0^t f(y)g(t-y)e^{-st}\,dy\,dt. \end{aligned}$$

The integration region is

$$\{(t,y) : 0 \le t < \infty,\, 0 \le y \le t\} = \{(t,y) : 0 \le y < \infty,\, t \ge y\}$$

so interchanging the order and then shifting $\overline{t} = t - y$ we get

$$\begin{aligned} \mathcal{L}[f*g] &= \int_0^\infty f(y) \int_y^\infty g(t-y)e^{-st}\,dt\,dy \\ &= \int_0^\infty f(y) \int_0^\infty g(\overline{t})e^{-s(y+\overline{t})}\,d\overline{t}\,dy \\ &= \left(\int_0^\infty f(y)e^{-sy}\,dy \right) \left(\int_0^\infty g(\overline{t})e^{-s\overline{t}}\,d\overline{t} \right) \\ &= \mathcal{L}[f]\mathcal{L}[g]. \end{aligned}$$

For (iv); Change variables with $y = s^{1/2}t^{1/2}$ (and $dt = \frac{2}{s}y\,dy$) to get

$$\mathcal{L}[t^{1/2}] = \int_0^\infty t^{1/2}e^{-st}\,dt = \frac{1}{s^{3/2}} \int_0^\infty y\left(2ye^{-y^2}\right)dy = \frac{\sqrt{\pi}}{2s^{3/2}}$$

after calculating the (standard) integral $\int_0^\infty y \cdot (2ye^{-y^2})\,dy = \int_0^\infty e^{-y^2}\,dy$ by parts. ∎

2. Solving LCC IVPs

The Laplace transform can be used to solve LCC initial value problems. The method is particularly useful if the forcing is piecewise defined or contains δ's, since the transforms are nice. Take, for example, the second-order equation

$$au'' + bu' + cu = f(t), \quad u(0) = p,\ u'(0) = q. \tag{2.1}$$

Set $U = \mathcal{L}[u]$ and $F = \mathcal{L}[f]$. Take the Laplace transform of the ODE to get

$$a(-u'(0) - su(0) - s^2U) + b(-u(0) + sU) + cU = F(s).$$

Now use the initial conditions to plug in fo $u(0)$ and $u'(0)$, yielding

$$\begin{aligned} U(s) &= \frac{1}{as^2 + bs + c}\left(F(s) + au'(0) + (as + b)u(0)\right) \\ &= H(s)F(s) + B(s) \end{aligned}$$

$$\text{where } H(s) = \frac{1}{as^2 + bs + c}, \quad B(s) = \cdots .$$

The function H is the **transfer function** for the system. Note that the ICs only matter for the other term, $B(s)$. Taking the inverse transform and using the convolution rule,

$$u(t) = h * f + b = \int_0^t f(t_0)h(t - t_0)\,dt_0 + b(t)$$

where $h = \mathcal{L}^{-1}(H)$ is the Green's function for the system. The function $h(t)$ is the solution to

$$au'' + bu' + cu = \delta(t - t_0), \quad u(0) = u'(0) = 0.$$

That is, $h(t)$ is the response of the system to a unit forcing applied at $t = t_0$.

Important note: We do **not** write the transformed solution as

$$U(s) = H(s)(F(s) + C(s)) \implies u = h * f + h * c$$

even though it is tempting to use the convolution theorem on both terms. The reason is that factoring out $H(s)$ in the second term leads to a function that fails the 'nice decay' test:

$$\lim_{s\to\infty} C(s) \neq 0.$$

Thus $\mathcal{L}^{-1}(C)$ cannot be a function (it is, in fact, a distribution), so writing the solution this way creates a mess. For this reason, the 'initial condition' part is typically inverted directly.

2.1. Straightforward example.

The Laplace transform is used to solve

$$u'' - 2u' = f(t), \quad u(0) = 1, \quad u'(0) = -2.$$

Take the Laplace transform to get (with $U = \mathcal{L}(u)$)

$$(s^2U - su(0) - u'(0)) - 2(sU - u(0)) = F.$$

Rearrange and solve for U, plugging in for the ICs to get

$$\begin{aligned} U &= \frac{1}{s^2 - 2s}\left(F(s) + s - 4\right) \\ &= H(s)F(s) + B(s), \end{aligned}$$

where

$$H(s) = \frac{1}{s^2 + bs + c}, \quad B(s) = (s - 4)H(s).$$

The function $H(s)$ is the **transfer function** for this IVP. Note that $\lim_{s\to\infty} s-4 \neq 0$ so by the 'nice decay' rule (1.4), $\mathcal{L}^{-1}(s-4)$ is not nice; we must invert $B(s)$ directly.
Now apply the convolution rule to the first term to get the solution

$$u(t) = h * f + \mathcal{L}^{-1}(B) = \int_0^\infty h(t-t_0)f(t_0)\,dt_0 + b(t)$$

To find the inverse transforms, use **partial fractions** (see other resources for this) to break up into fractions that can be inverted using the basic rules:

$$H(s) = \frac{1}{s(s-2)} = \frac{c_1}{s} + \frac{c_2}{s-2} \implies H(s) = -\frac{1/2}{s} + \frac{1/2}{s-2},$$

$$B = \frac{s-4}{s(s-2)} = \frac{c_3}{s} + \frac{c_4}{s-2} \implies B = \frac{2}{s} - \frac{1}{s-2}$$

It follows from the exponential rule $e^{at} \to 1/(s-a)$ that

$$h = -\frac{1}{2} + \frac{1}{2}e^{2t}, \quad b(t) = 2 - e^{2t}.$$

Thus the solution is, explicitly,

$$u(t) = \int_0^\infty \frac{1}{2}(e^{2(t-t_0)} - 1)f(t_0)\,dt_0 + 2 - e^{2t}.$$

This is the solution you would get by using variation of parameters for ODE IVPs.

Complex variables insight: It is worth computing the inverse transform for H directly:

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(s-2)}\,ds \text{ with } c > 2.$$

By the p/q' rule for poles, each simple residue at z_0 gives an exponential (e^{st} at $s = z_0$), e.g.

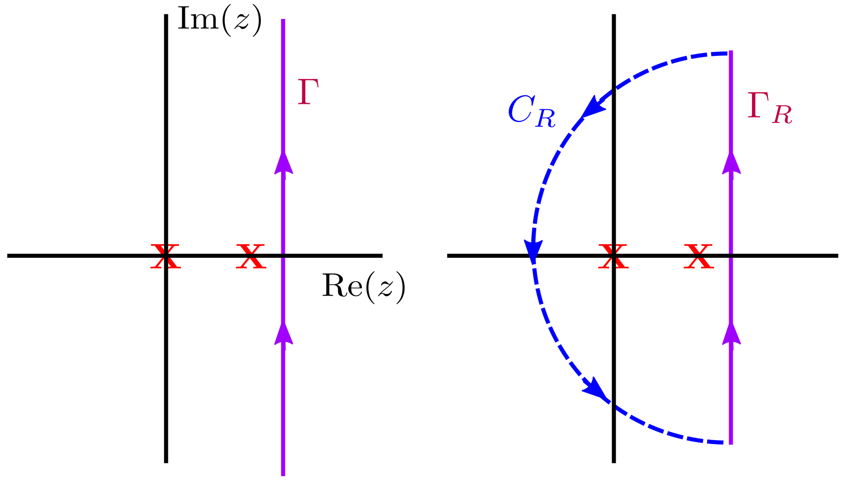
$$\operatorname{Res}(f(s);2) = \frac{e^{2t}}{2}.$$

This justifies the need to put the contour to the right of all singularities and the observation that exponentials in t correspond to poles of the transform.

To evaluate the contour integral properly, we need to close it. In this case, the left semi-circle of radius R (taking $R \to \infty$) is correct:

$$s = R\cos\theta + iR\sin\theta \implies |e^{st}| = e^{Rt\cos\theta} \leq 1 \text{ iff } \theta \in [\pi/2, 3\pi/2].$$

The left semi-circle works since t **is always positive**. This is good, because we want it to contain all the poles to the left of the vertical line.



The rest is standard. Assume $t > 0$ and let

$$I = \int_{c-i\infty}^{c+i\infty} f(s)\,ds, \qquad f(s) = \frac{1}{2\pi i}\frac{e^{st}}{s(s-2)}.$$

Now that the semi-circle has been checked to be good, estimate

$$\Big|\int_{C_R} f(s)\,ds\Big| \le \frac{1}{2\pi}\pi R \frac{1}{R^2 - 2R} \to 0 \text{ as } R \to \infty.$$

It follows that for the closed contour $\Gamma_R + C_R$,

$$\oint f(s)\,ds \to h(t) \text{ as } R \to \infty.$$

Now apply the residue theorem. Both residues are inside the semi-circle for large R, so

$$h(t) = \oint f(s)\,ds = \frac{e^{st}}{2s-2}\Big|_{s=2} + \frac{e^{st}}{2s-2}\Big|_{s=0} = \frac{1}{2}e^{2t} - \frac{1}{2}.$$

Note that a different argument would be needed if $t > 0$, but the Laplace transform setup assumes positive times so we don't need to consider it.

2.2. **Technical note (Jordan curve lemma):** Earlier, we used the ML estimate to deal with the semi-circle contours. This crude estimate is not always enough. Take, for instance,

$$\mathcal{L}^{-1}(1/s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s}e^{st}\,ds. \tag{2.2}$$

Let C_R be the 'good' (left) semi-circle and let $f(s) = e^{st}/s$. Ideally, $\int_{C_R}$ should vanish so

$$\oint f(s)\,ds = \int_{c-i\infty}^{c+i\infty} f(s)\,ds + \int_{C_R} f(s)\,ds \to 2\pi i\mathcal{L}^{-1}(1/s)$$

However, the $1/s$ decay is not fast enough for the ML estimate, since

$$\left|\int_{C_R} f(z)\,dz\right| \leq (\pi R)\frac{1}{R} = \pi$$

which does not vanish as $R \to \infty$. An improved estimate is needed using the fact that e^{st} is 'mostly' small on the semi-circle.

Jordan curve lemma (JCL): Let C_R be the 'good' semi-circle for an integral

$$\int_{C_R} e^{ikz} f(z)\,dz$$

i.e. upper half for $k > 0$, left for $ik = t > 0$ and so on. Then

$$\left|\int_{C_R} e^{ikz} f(z)\,dz\right| \leq \frac{\pi M}{|k|}, \qquad M = \text{ max value of } |f| \text{ on } C_R. \tag{2.3}$$

If $k < 0$ the same is true, but for the lower half semi-circle. Informally,

$$\int_{C_R} |e^{iz}|\,dz \leq \frac{\text{const.}}{R}.$$

That is, the e^{iz} part gives an extra factor of $1/R$ on the good semi-circle.

Using this estimate, we can evaluate (2.2). Let C_R be the left-half semi-circle, which was previously shown to be good ($|e^{iz}| \leq 1$ on C_R). By the Jordan curve lemma,

$$\left|\int_{C_R} \frac{e^{st}}{s}\,ds\right| \leq \frac{\pi}{tR} \to 0 \text{ as } R \to \infty.$$

Informally, this would read

$$\left|\int_{C_R} \frac{e^{st}}{s}\,ds\right| \leq \underbrace{\frac{C}{R}}_{\text{JCL}}\underbrace{(\pi R)}_{L}\underbrace{\frac{1}{R}}_{M} = \frac{C}{R}.$$

From here, integrate over the closed contour to get

$$2\pi i \sum \text{Res} = \oint f(s)\,ds \to \int_{c-i\infty}^{c+i\infty} f(s)\,ds \text{ as } R \to \infty.$$

There is one residue (simple) at $z_0 = 0$ so

$$\mathcal{L}^{-1}(1/s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s}e^{st}\,ds = \text{Res}(f, 0) = 1.$$

2.3. **Example (resonance):** Consider the forced oscillator

$$u'' + u = \sin \omega t, \quad u(0) = u'(0) = 0.$$

The transform of the solution (after computing the transform of the RHS directly) is

$$U(s) = \frac{1}{s^2+1} \cdot \frac{\omega}{\omega^2 + s^2}.$$

We can inverse transform and use the standard rules to simplify. But consider the inverse transform formula

$$u(t) = \frac{1}{2\pi i} \int_\Gamma U(s) e^{st}\, ds = \frac{\omega}{2\pi i} \int_\Gamma \frac{e^{st}}{q(s)}\, ds,$$
$$q(s) := (s-i)(s+i)(s-\omega i)(s+\omega i),$$

where $\Gamma = \{z(t) : c + it,\ t \in (-\infty, \infty)\}$ is the vertical line shifted to the right of all singularities (so $c > 1$ and $c > \omega$). Note that

$$|q(s)| \sim R^4, \quad R \to \infty$$

so Γ can be closed with the left semi-circle C_R as before, where

$$|e^{st}| = e^{Rt\cos\theta} \le 1 \text{ (for } \theta \in [\pi/2, 3\pi/2]).$$

Thus

$$\oint = \int_\Gamma + \int_{C_R} \to \int_\Gamma \text{ as } R \to \infty.$$

The poles are at the zeros s_k of the denominator $q(s)$ and so

$$u(t) = \omega \sum_k \operatorname{Res}\left(\frac{e^{st}}{q(s)}, s_k \right).$$

Note that these poles are the zeros of the char. polynomial and the poles of the forcing.

We can now deduce the response from the residues. If the pole is simple,

$$\operatorname{Res} = e^{s_k t}/q'(s_k) = Ce^{s_k t} \implies \text{oscillation at freq } \operatorname{Im} s_k$$

since the s_k's are purely imaginary. However, if the pole is order 2, one can show

$$\operatorname{Res} = Cte^{s_k t} \implies \text{resonance}.$$

That is, the order of the poles tells us whether resonance occurs. Note that a pole of order 2 occurs precisely when **the poles of the forcing and the zeros of the char. poly overlap**. Thus, from a plot of poles in the complex plane, one can read off the behavior.

3. PDEs

The **Laplace** transform can be used to solve time dependent PDEs in the **time** domain $t \in [0, \infty)$. This includes the heat equation in $x \in [0, \infty)$ and even in a bounded interval $x \in [a, b]$. A a typical example, consider[2]

$$\begin{aligned} u_t &= u_{xx},\ \ x \in (0,\infty),\ \ t > 0 \\ u(0,t) = f(t),\ u(x,t) &\text{ bounded },\ \ t > 0 \\ u(x,0) &= 0 \end{aligned} \tag{3.1}$$

Physical scenario: A half-infinite container is initially empty, and then some material is introduced at the left side and diffuses (e.g. discarded toxic waste seeping into a lake).
We can solve by taking a sine transform in space (take $\langle \cdot, \sin kx \rangle$), leaving ODEs in t.
Instead, we take the Laplace transform in **time**, leaving ODEs **in space**.

Start by taking $\mathcal{L}$ of the PDE (in t) to get

$$\mathcal{L}[u_t] = \mathcal{L}[u_{xx}]$$
$$\implies \int_0^\infty u_t e^{-st}\, ds = \int_0^\infty u_{xx} e^{-st}\, ds.$$

Use the derivative rule on the left side and factor out the x-derivatives on the right side:

$$-u(x,0) + sU = U_{xx}.$$

Now plug in the IC $u(x,0) = 0$ to get

$$U_{xx} - sU = 0.$$

Finally, transform the BC at $x = 0$ (it's a function of t!) to get

$$U(0,t) = \mathcal{L}[f(t)] = F(s).$$

Thus, the ODE problem to solve for $U(x,s)$ (as a function of x) is

$$U_{xx} - sU = 0, \quad U(0,s) = F(s).$$
$$\implies U = c_1(s)e^{-\sqrt{s}x} + c_2(s)e^{\sqrt{s}x}, \quad x > 0.$$

Now observe that if $u(x,t)$ is to be bounded, then

$$|U(x,s)| \le \int_0^\infty |u(x,t)|e^{-st}\, ds \le \frac{C}{s}.$$

Thus, $u(x,t)$ bounded also implies that $U(x,s)$ must be bounded (in fact, slightly better), so we must exclude the $e^{\sqrt{s}x}$ term. Using this and $U(0,s) = F(s)$,

$$U(x,s) = F(s)e^{-\sqrt{s}x} = F(s)H(x,s).$$

Now take the inverse Laplace transform and use a table to look up $\mathcal{L}^{-1}$ of H (a non-obvious but standard transform; see Ex. 13.2.9 of Haberman for the calculation) to get

$$u(x,t) = \int_0^t f(t_0)h(x,t-t_0)\, dt_0, \qquad h(x,t) = \frac{1}{\sqrt{4\pi}} \frac{x}{t^{3/2}} e^{-x^2/4t}.$$

This function $h(x,t)$ is a Green's function for the boundary input $u(0,t)$ in the domain $[0,\infty)$. Note that the convolution is in t, not in x.

[2]Example borrowed from J. Logan, *Applied Partial Differential Equations, Ch. 2.*

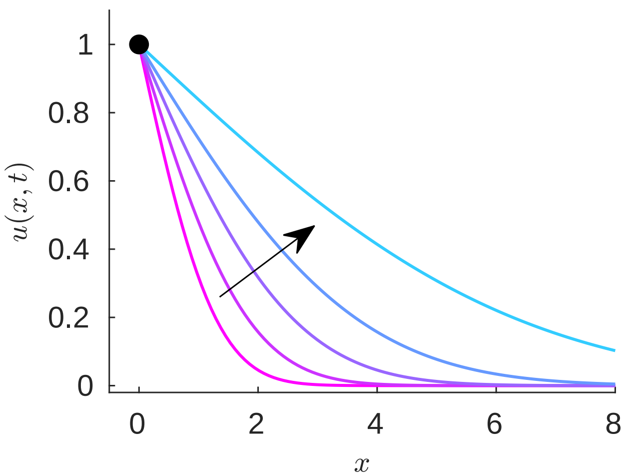
Special case: Suppose

$$u(0,t) = 1.$$

Then $F(s) = 1/s$ and the inversion can be done directly (or plug into the convolution):

$$U(x,s) = \frac{1}{s}e^{-\sqrt{s}x} \implies u(x,t) = 1 - \operatorname{erf}(\frac{x}{2\sqrt{t}}) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{t}} e^{-\xi^2}\, d\xi.$$

Since $\operatorname{erf}(x) \to 1$ as $x \to \infty$, $u(x,t) \to 0$ as expected (satisfies the BC at ∞). Moreoever, the solution is only a function of $x/\sqrt{t}$, giving it a self-similar shape (see **??**).



Remark (comparison): Notice that the Fourier transform **separates in space**, leading to independent ODEs in t. Thus, it 'separates' ICs like

$$u(x,0) = f(x) \implies F(k)$$

which makes non-zero ICs easy to deal with.
On the other hand, the Laplace transform **separates in time** and separates the BCs:

$$u(0,t) = f(t) \implies F(s)$$

The Laplace transform is useful here because the BC is the only non-zero input.

Connection to fundamental solution: The similarity of h to the fundamental solution

$$G(x,t) := \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$$

is not coincidential. Recall that we found the Green's function for the heat equation in $[0,\infty)$ with initial condition $u(x,0) = f9x)$ was

$$\tilde{G}(x,x_0,t) = G(x - x_0, t) - G(x + x_0, t).$$

Then by a quick calculation, we find that

$$h = -\frac{\partial \tilde{G}}{\partial x}\Big|_{x_0=0}.$$

This relates the response of the system to ICs ($\tilde{G}$) and to BCs (h). We saw a similar relation in solving inhomogeneous BVPs with Green's functions, and is a recurring pattern.

3.1. **Example: Transport equation.** The Laplace transform is used to solve the transport equation (with no diffusion!)

$$\begin{aligned} u_t + cu_x &= 0, \quad x \in (0, \infty), \\ u(0,t) &= f(t) \\ u(x,0) &= g(x) \end{aligned}$$

This fundamental equation describes transport of a quantity $u(x,t)$ at a speed c, e.g. the density of cars in free flowing traffic on a highway (without any traffic jams!). Suppose $g(x) = 0$ for simplicity. Take the Laplace transform of the PDE and BC to get

$$sU(x,s) - u(x,0) + cU_x(x,s) = 0 \implies sU + cU_x = 0,$$
$$U(0,s) = F(s).$$

Note that the BC (a function of time) is transformed here. We have that U solves

$$cU_x + sU = 0, \quad U(0,s) = F(s).$$

Solve this first order linear ODE in x to get

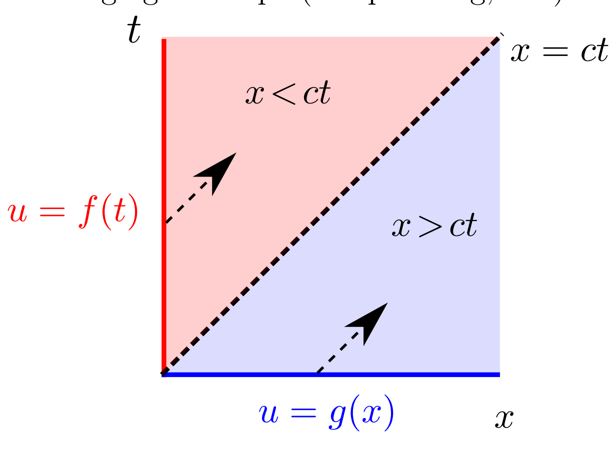
$$U(x,s) = e^{-(x/c)s}F(s)$$

Now inverse transform using the shift theorem to get

$$u(x,t) = f(t - x/c)H(ct - x) = \begin{cases} f(t - x/c) & x < ct \\ 0 & x > ct \end{cases}.$$

The equation carries information at a speed c from the two boundaries (at $t = 0$ and $x = 0$). The IC defines the solution in $\{x > ct\}$ and the BC defines it in $\{x < ct\}$.
The solution is **constant along the 'characteristics'** $x = ct + \text{const.}$.

This is typical of the transport equation (and the transport term cu_x), which propagates u at a speed of c without changing its shape (no spreading, etc.).



Remark: The best way to solve this problem is with the **method of characteristics** (beyond the scope of the course), which is used for PDEs with waves that propagate - the result above hints at the important structure (a 'function of $x - ct$'). The idea is to find the characteristics directly and use the fact that information propagates along them.