

Math 5510/Math 4510 - Partial Differential Equations: Heat conduction in higher dimensions

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Outline

- 1 Heat Equation 3D
 - Derivation
 - Heat Equation

- 2 Laplacian in Other Coordinates
 - Poisson's and Laplace's Equations
 - Other Coordinates

Heat Conduction in a Higher Dimensions

Previously we developed the *heat equation* for a one-dimensional rod

We want to extend the *heat equation* for higher dimensions

Conservation of Heat Energy: In any volume element, the basic conservation equation for **heat** satisfies

Rate of change of heat energy in time	=	Heat energy flowing across boundaries per unit time	+	Heat energy generated inside per unit time
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Define $c(x, y, z)$ to be the *specific heat of a material* (the *heat energy* required to raise a unit mass of a material a unit of temperature)

Define $\rho(x, y, z)$ to be the *mass density* (per unit volume)

Define $u(x, y, z, t)$ as the *temperature of a material*

Heat Conduction in a Higher Dimensions

The **specific heat**, c , **mass density**, ρ , and **temperature**, u , are used with the conservation law above to create the general **heat equation**

The total energy in a volume element R satisfies:

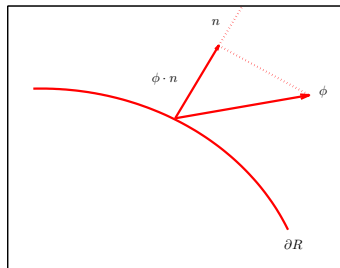
$$\text{Total Energy} = \iiint_R c\rho u \, dV.$$

The rate of change of **heat energy** in time is given by

$$\frac{d}{dt} \iiint_R c\rho u \, dV.$$

Heat Conduction in a Higher Dimensions

Define $\phi(x, y, z, t)$ as the *heat flux vector* for the heat crossing the surface of the region R denoted ∂R , and define n as the *outward normal vector*



By convention the heat flux is the flow directed into the region R , so the *heat flux* into the region R is the integral over ∂R of $-\phi \cdot n$.

$$-\iint_{\partial R} \phi \cdot n \, dS$$

Heat Conduction in a Higher Dimensions

Define $Q(x, y, z, t)$ as the *heat energy* generated per unit time from the *sources* or *sinks* inside the region R .

This gives

$$\iiint_R Q \, dV.$$

The **Conservation of Heat Energy** combines these terms to give:

$$\frac{d}{dt} \iiint_R c\rho u \, dV = - \oiint_{\partial R} \phi \cdot n \, dS + \iiint_R Q \, dV.$$

We need to combine these terms to obtain the general **Heat Equation**.

Heat Conduction in a Higher Dimensions

Theorem (Divergence or Gauss's Theorem)

Suppose R is a subset of \mathbb{R}^3 , which is compact and has a piecewise smooth boundary ∂R . If ϕ is a continuously differentiable vector field defined on a neighborhood of R , then we have:

$$\oiint_{\partial R} (\phi \cdot n) dS = \iiint_R (\nabla \cdot \phi) dV.$$

The **Conservation of Heat Energy** combines these terms to give:

$$\frac{d}{dt} \iiint_R c\rho u dV = - \iiint_R (\nabla \cdot \phi) dV + \iiint_R Q dV.$$

Heat Conduction in a Higher Dimensions

The previous equation is rearranged to give:

$$\iiint_R \left(c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q \right) dV = 0.$$

Since this holds for any region R , we have the **heat equation**:

$$c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \phi + Q.$$

Fourier's law of heat conduction satisfies:

$$\phi = -K_0 \nabla u,$$

which produces the **heat equation** in higher dimensions:

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q.$$

Heat Equation in a Higher Dimensions

The **heat equation** in higher dimensions is:

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q.$$

If the Fourier coefficient is constant, K_0 , as well as the specific heat, c , and material density, ρ , and if there are no sources or sinks, $Q \equiv 0$, then the **heat equation** becomes

$$\frac{\partial u}{\partial t} = k \nabla^2 u, \quad t > 0 \quad \text{and} \quad (x, y, z) \in R,$$

where $k = K_0/(c\rho)$ and

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

in Cartesian coordinates.

Poisson's and Laplace's Equations

The **heat equation** in higher dimensions is:

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q.$$

If the Fourier coefficient is constant, K_0 , then the **Steady-State** problem can be written:

$$\nabla^2 u = -\frac{Q}{K_0},$$

which is **Poisson's equation**

Furthermore, if there are **no sources** or **sinks** ($Q \equiv 0$), then we obtain **Laplace's equation**

$$\nabla^2 u = 0.$$

Laplacian in 2D

In Cartesian coordinates, the Laplacian in 2D is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Recall that in polar coordinates

$$x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta).$$

By using the **chain rule** and the **dot product**, we find:

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

Laplacian in 3D

In cylindrical coordinates

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad \text{and} \quad z = z,$$

so

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

In spherical coordinates

$$x = \rho \sin(\phi) \cos(\theta), \quad y = \rho \sin(\phi) \sin(\theta), \quad \text{and} \quad z = \rho \cos(\phi),$$

so it can be shown (HW exercise):

$$\nabla^2 u = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin(\phi)} \frac{\partial}{\partial \phi} \left(\sin(\phi) \frac{\partial u}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2(\phi)} \frac{\partial^2 u}{\partial \theta^2}.$$