Calculus for the Biological Sciences

Exponential growth and decay

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Exponential growth and decay

Examples

Many quantities in nature change according to an exponential growth or decay function of the form $P = P_0 e^{kt}$, where P_0 is the initial quantity and k is the continuous growth or decay.

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- Find t so that: $0.6 = 2.4e^{-0.004t}$

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- $39 = 19.5 e^{k \cdot 25}$

Doubling time and Half-life

Definition

The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.
The **half-life** of an exponentially decaying quantity is the time required for the quantity reduced by a factor of one half.

Examples

Harder problem: Show that every exponentially increasing function has a fixed doubling time.

- **Problem 3.** The release of chlorofluorocabons (CFC) used in air conditioners and household sprays destroys the ozone in the upper atmosphere. The quantity of ozone, Q, is decaying exponentially at a continuous rate of 0.25% per year. What is the half-life of ozone?

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- $t = \frac{\ln(1/2)}{-0.0025} = 277$

- **Problem 4.** If \$100,000 is deposited in an account paying interest at a rate of 5% per year, compounded continuously, the how long does it take for the balance in the account to reach \$150,000?

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- We need to find t so that: $150,000 = 100,00e^{0.05t}$.

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- **Extra question:** Calculate the doubling time, D, for interest rates of 1%, 3%, 5%, and 6% per year, compounded continuously.
- Assume the doubling time corresponding to the interest rate i% per year is D_i. Use the results in the previous part to compare D_i and $70/i$.

Rule of 70

To compute the approximate doubling time of an investment, divide 70 by the percent annual interest rate.

Many business deals involve payments in the future. For example, when a car is bought on credit, payments are made over a period of time. Being paid \$1000 in the future is worse than being paid \$1000 today. Therefore, even without considering inflation, if we are to accept payment in the future, we would expect to be paid more to compensate for this loss of potential earnings. The question is "How much more?".

Definition

- The **future value**, B, of a payment P, is the amount to which the P would have grown if deposited today in an interest-bearing bank account.
- The **present value**, P, of a future payment B, is the amount that would have to be deposited in a bank account today to produce exactly B in the account at the relevant time in future.

Present and future values

Suppose B is the future value of P, and P is the present value of B.

- If interest is compounded annually at a rate r for t years, then

$$B = P(1+r)^t, \qquad \text{or equivalently,} \qquad P = \frac{B}{(1+r)^t}.$$

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- If interest is compounded continuously at a rate r for t years, then

$$B = Pe^{rt}, \qquad \text{or equivalently,} \qquad P = \frac{B}{e^{rt}} = Be^{-rt}.$$

Example

- **Problem 5.** You win the lottery and are offered the choice between $1 million in four yearly installments of $250,000 each, starting now, and a lump-sum payment of $920,000 now. Assuming a 6% interest rate per year, compounded continuously, and ignoring taxes, which should you choose?

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- **Problem 5.** You win the lottery and are offered the choice between \$1 million in four yearly installments of \$250,000 each, starting now, and a lump-sum payment of \$920,000 now. Assuming a 6% interest rate per year, compounded continuously, and ignoring taxes, which should you choose?
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- (a) Compare present value of the first payment method.
- (b) Compare future value of two payment methods.
- (c) Find the answer if the interest is compounded annually at a rate 5% .

Compare future values

- $250,000 + 250,000e^{(-0.06)(1)} + 250,000e^{(-0.06)(2)} + 250,000e^{(-0.06)(3)}$

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- $250,000 + 250,000e^{-(0.06)(1)} + 250,000e^{-(0.06)(2)} + 250,000e^{-(0.06)(3)}$
- 250,000+235,441+221,730+208,818=915,989

Compare present values

- $920,000e^{(0.06)(3)} = 1,101,440$

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- $920,000e^{(0.06)(3)} = 1,101,440$
- $250,000e^{(0.06)(3)} + 250,000e^{(0.06)(2)} + 250,000e^{(0.06)(1)} + 250,000$

Compare present values

- $920,000e^{(0.06)(3)} = 1,101,440$
- $250,000e^{(0.06)(3)} + 250,000e^{(0.06)(2)} + 250,000e^{(0.06)(1)} + 250,000$
- 299,304+281,874+265,459+250,000=1,096,637

Compare present values

- $250,000 + \frac{250{,}000}{(1.05)} + \frac{250{,}000}{(1.05)^2} + \frac{250{,}000}{(1.05)^3}$

Compounded annually

- $250,000 + \frac{250{,}000}{(1.05)} + \frac{250{,}000}{(1.05)^2} + \frac{250{,}000}{(1.05)^3}$
- $250,000 + 238,095 + 226,757 + 215,959$