Calculus for Biological Sciences

Lecture Notes – Functions and Change

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Outline

1. Definitions and Properties of Functions
   - Definition of a Function
   - Vertical Line Test
   - Function Operations
   - Composition of Functions
   - Even and Odd Functions
   - One-to-One Functions
   - Inverse Functions

Definitions and Properties of Functions

Definitions and Properties of Functions

- Functions form the basis for most of this course
- A function is a relationship between one set of objects and another set of objects with only one possible association in the second set for each member of the first set

Definition of a Function

Definition: A **function** of a variable x is a rule f that assigns to each value of x a unique number $f(x)$. The variable x is the **independent variable**, and the set of values over which x may vary is called the **domain** of the function. The set of values $f(x)$ over the domain gives the **range** of the function

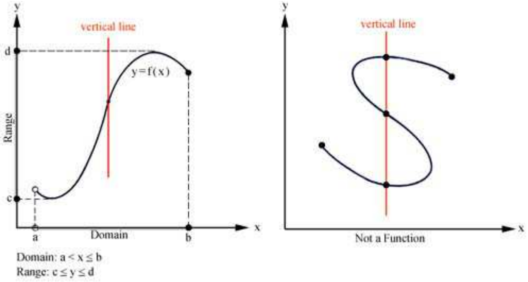
Definition of a Graph

Definition: The **graph of a function** is defined by the set of points (x, y) such that $y = f(x)$, where f is a function.

- Often a function is described by a **graph** in the xy-coordinate system
- By convention x is the domain of the function and y is the range of the function
- The **graph** is defined by the set of points $(x, f(x))$ for all x in the domain

Vertical Line Test

The **Vertical Line Test** states that a curve in the xy-plane is the graph of a function if and only if each vertical line touches the curve *at no more than one point*



Example of Domain and Range

Example 1: Consider the function

$$f(t) = t^2 - 1$$

Skip Example

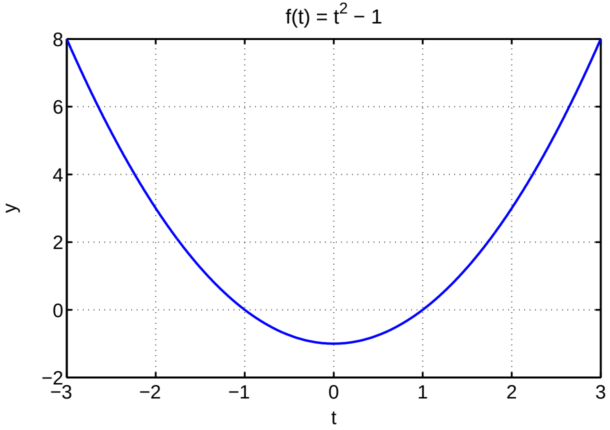
a. What is the range of $f(t)$ (assuming a domain of all t)?

Solution a: $f(t)$ is a parabola with its vertex at $(0, -1)$ pointing up.

Since the vertex is the low point of the function, it follows that range of $f(t)$ is $-1 \leq y < \infty$

Graph of Example 1

Graph for the domain and range of $f(t)$



Example of Domain and Range

Example 1 (cont): More on the function

$$f(t) = t^2 - 1$$

b. Find the domain of $f(t)$, if the range of f is restricted to $f(t) < 0$

Solution b: Solving $f(t) = 0$ gives $t = \pm 1$

It follows that the domain is $-1 < t < 1$

Addition and Multiplication of Functions

Example 2: Let $f(x) = x - 1$ and $g(x) = x^2 + 2x - 3$

Skip Example

Determine $f(x) + g(x)$ and $f(x)g(x)$

Solution: The addition of the two functions

$$f(x) + g(x) = x - 1 + x^2 + 2x - 3 = x^2 + 3x - 4$$

The multiplication of the two functions

$$\begin{aligned} f(x)g(x) &= (x-1)(x^2+2x-3) \\ &= x^3 + 2x^2 - 3x - x^2 - 2x + 3 \\ &= x^3 + x^2 - 5x + 3 \end{aligned}$$

Addition of Function

Example 3: Let

$$f(x) = \frac{3}{x-6} \quad \text{and} \quad g(x) = -\frac{2}{x+2}$$

Skip Example

Determine $f(x) + g(x)$

Solution: The addition of the two functions

$$\begin{aligned} f(x) + g(x) &= \frac{3}{x-6} + \frac{-2}{x+2} = \frac{3(x+2) - 2(x-6)}{(x-6)(x+2)} \\ &= \frac{x+18}{x^2 - 4x - 12} \end{aligned}$$

Composition of Functions

Composition of Functions is another important operation for functions

Given functions $f(x)$ and $g(x)$, the composite $f(g(x))$ is formed by inserting $g(x)$ wherever x appears in $f(x)$

Note that the domain of the composite function is the range of $g(x)$

Composition of Functions

Example 4: Let

$$f(x) = 3x + 2 \quad \text{and} \quad g(x) = x^2 - 2x + 3$$

Skip Example

Determine $f(g(x))$ and $g(f(x))$

Solution: For the first composite function

$$f(g(x)) = 3(x^2 - 2x + 3) + 2 = 3x^2 - 6x + 11$$

The second composite function

$$g(f(x)) = (3x + 2)^2 - 2(3x + 2) + 3 = 9x^2 + 6x + 3$$

Clearly, $f(g(x)) \neq g(f(x))$

Even and Odd Functions

A function f is called:

1. **Even** if $f(x) = f(-x)$ for all x in the domain of f. In this case, the graph is symmetrical with respect to the y-axis

2. **Odd** if $f(x) = -f(-x)$ for all x in the domain of f. In this case, the graph is symmetrical with respect to the origin

Example of Even Function

Consider our previous example

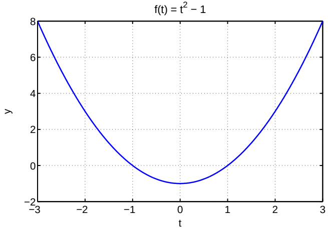
$$f(t) = t^2 - 1$$

Since

$$f(-t) = (-t)^2 - 1 = t^2 - 1 = f(t),$$

this is an even function.

The Graph of an Even Function is symmetric about the y-axis



One-to-One Function

Definition: A function f is **one-to-one** if whenever $x_1 \neq x_2$ in the domain, then $f(x_1) \neq f(x_2)$.

Equivalently, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Inverse Functions

Definition: If a function f is **one-to-one**, then its corresponding **inverse function**, denoted f^{-1}, satisfies:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Since these are composite functions, the domains of f and f^{-1} are restricted to the ranges of f^{-1} and $f(x)$, respectively

Example of an Inverse Function

Consider the function

$$f(x) = x^3$$

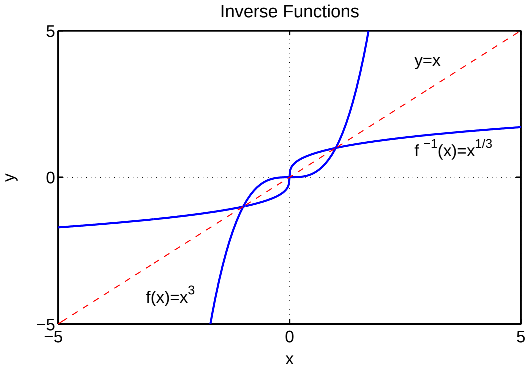
It has the inverse function

$$f^{-1}(x) = x^{1/3}$$

The domain and range for these functions are all of x

$$f^{-1}(f(x)) = \left(x^3\right)^{1/3} = x = \left(x^{1/3}\right)^3 = f(f^{-1}(x))$$

Example of an Inverse Function



These functions are mirror images through the line $y = x$ (**the Identity Map**)