Section 1.8: Limits of Functions Using Numerical and Graphical Techniques

Definition: Let f be defined on an open interval containing the number c except possibly at c. Then

$$\lim_{x \to c} f(x) = L$$

provided the value of $f(x)$ can be made arbitrarily close to the number L by taking x sufficiently close to c but not equal to c.

One Sided Limits

Left Hand Limit: We write

$$\lim_{x \to c^-} f(x) = L_L$$

and say *the limit as x approaches c from the left of $f(x)$ equals L_L provided we can make $f(x)$ arbitrarily close to the number L_L by taking x sufficiently close to, but less than c.*

Right Hand Limit: We write

$$\lim_{x \to c^+} f(x) = L_R$$

and say *the limit as x approaches c from the right of $f(x)$ equals L_R provided we can make $f(x)$ arbitrarily close to the number L_R by taking x sufficiently close to, but greater than c.*

Observations

Observation 1: The limit L of a function $f(x)$ as x approaches c does not depend on whether $f(c)$ exists or what it's value may be.

Observation 2: If $\lim_{x\to c} f(x) = L$, then the number L is unique. That is, a function can not have two different limits as x approaches a single number c.

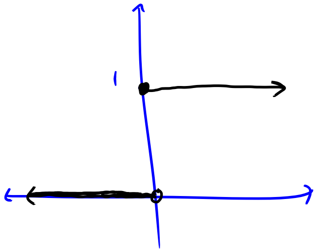
Observation 3: A function need not have a limit as x approaches c. If $f(x)$ can not be made arbitrarily close to any one number L as x approaches c, then we say that $\lim_{x\to c} f(x)$ **does not exist** (shorthand **DNE**).

A Limit Failing to Exist

H is the Heaviside step function

Consider $H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$. Evaluate if possible

$$\lim_{x\to 0^-} H(x), \qquad \lim_{x\to 0^+} H(x), \quad \text{and} \quad \lim_{x\to 0} H(x)$$



$\lim_{x\to 0^-} H(x) = 0$ from the graph

$\lim_{x\to 0^+} H(x) = 1$ from the graph

$\lim_{x\to 0} H(x)$ Does not exist (DNE)

There is no one number L that $H(x)$ values get arbitrarily close to.

This is called a jump when

$$\lim_{x\to c^-} f(x) = L_L \qquad \lim_{x\to c^+} f(x) = L_R$$

and $\quad L_L \neq L_R$

Weakness of Technology

Suppose we wish to investigate

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x^2}\right).$$

We consider values of x closer to zero, and plug them into a calculator. Let's look at two attempts.

x	$\sin\left(\frac{\pi}{x^2}\right)$
-0.1	0
-0.01	0
-0.001	0
0	undefined
0.001	0
0.01	0
0.1	0

limit looks like zero

x	$\sin\left(\frac{\pi}{x^2}\right)$
$-\frac{2}{3}$	0.707
$-\frac{2}{13}$	0.707
$-\frac{2}{23}$	0.707
0	undefined
$\frac{2}{23}$	0.707
$\frac{2}{13}$	0.707
$\frac{2}{3}$	0.707

limit looks like 0.707..

Weakness of Technology

In every interval containing zero, the graph of $\sin(\pi/x^2)$ passes through every y-value between -1 and 1 infinitely many times.

$\lim_{x\to 0} \sin\left(\frac{\pi}{x^2}\right)$ DNE

"by oscillation"

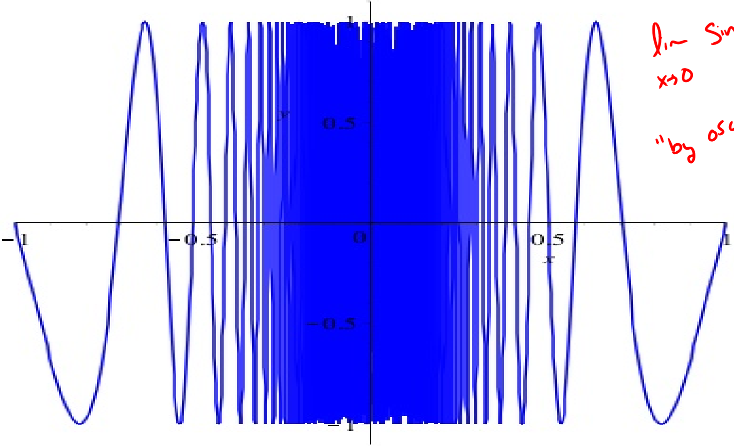


Figure: $y = \sin\left(\frac{\pi}{x^2}\right)$

Evaluating Limits

As this example illustrates, we would like to avoid too much reliance on technology for evaluating limits. The next section will be devoted to techniques for doing this for reasonably well behaved functions. We close with one theorem.

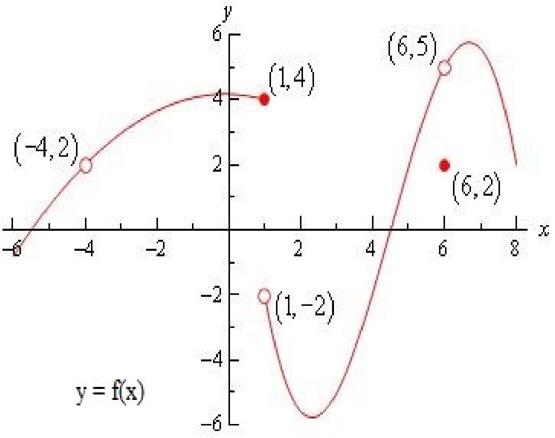
Theorem: Let f be defined on an open interval containing c except possible at c. Then

$$\lim_{x\to c} f(x) = L$$

if and only if

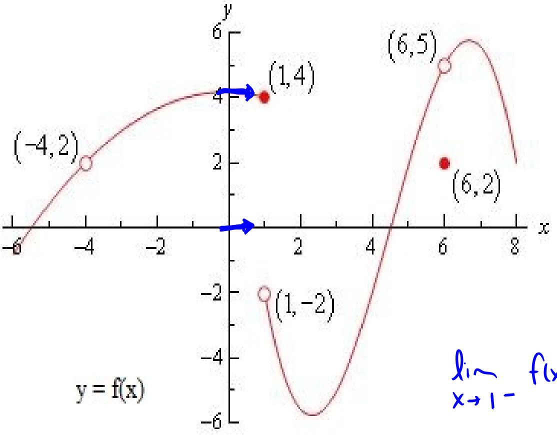
$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Limits from a graph



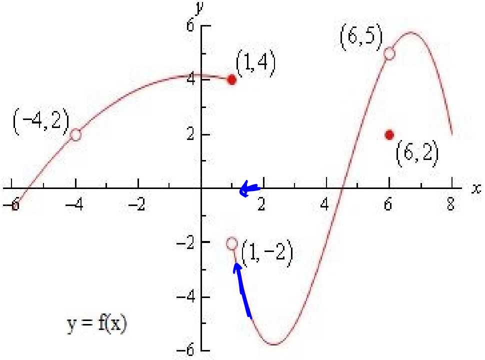
Evaluate $\lim_{x \to 1^-} f(x)$

Limits from a graph



$$\lim_{x\to 1^-} f(x) = 4$$

Question



aside: $\lim_{x\to 1} f(x)$ DNE

$$\lim_{x\to 1^+} f(x) =$$

(a) 4 (b) -2 (c) DNE (d) 1

Section 1.2: Limits of Functions Using Properties of Limits

We begin with two of the simplest limits we may encounter.

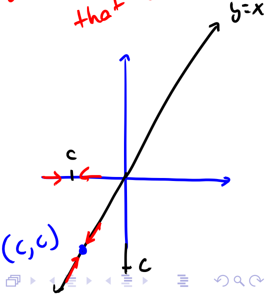
Theorem: If $f(x) = A$ where A is a constant, then for any real number c

$$\lim_{x\to c} f(x) = \lim_{x\to c} A = A$$

the limit of a constant is that constant

Theorem: If $f(x) = x$, then for any real number c

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c$$



Examples

(a) $\lim_{x \to 0} 7 = 7$

limit of a constant is that constant

(b) $\lim_{x \to \pi^+} 3\pi = 3\pi$

$\lim_{x \to c} f(x) = L$

if and only if

$\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$

We know that

$\lim_{x \to \pi} 3\pi = 3\pi$

(c) $\lim_{x \to -\sqrt{5}} x = -\sqrt{5}$

Since $\lim_{x \to c} x = c$

Question

$$\lim_{x \to 4^-} x =$$

$$\lim_{x \to 4} x = 4$$

(a) x

(b) -4

(c) 4

(d) the one sided limit can't be determined

Additional Limit Law Theorems

Suppose

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M, \quad \text{and } k \text{ is constant.}$$

Theorem: (Sums) $\lim_{x\to c}(f(x)+g(x)) = L+M$

Theorem: (Differences) $\lim_{x\to c}(f(x)-g(x)) = L-M$

Theorem: (Constant Multiples) $\lim_{x\to c} kf(x) = kL$

Theorem: (Products) $\lim_{x\to c} f(x)g(x) = LM$

Examples

Use the limit law theorems to evaluate if possible

(a) $\lim_{x\to 2}(3x+2)$ $= \lim_{x\to 2} 3x + \lim_{x\to 2} 2$ (sum)

$= 3\lim_{x\to 2} x + \lim_{x\to 2} 2$ (constant multiple)

$= 3\cdot 2 + 2 = 8$

Examples

Use the limit law theorems to evaluate if possible

(b) $\lim_{x\to -3}(x+1)^2$

Consider $\lim_{x\to -3}(x+1) = \lim_{x\to -3} x + \lim_{x\to -3} 1$

$= -3 + 1 = -2$

$\lim_{x\to -3}(x+1)^2 = \left(\lim_{x\to -3}(x+1)\right)\cdot\left(\lim_{x\to -3}(x+1)\right)$ (product)

$= (-2)\cdot(-2) = 4$

Examples

Use the limit law theorems to evaluate if possible

(c) $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} x+2, & x<0 \\ 1, & x=0 \\ 2x-3, & x>0 \end{cases}$

$\lim_{x\to 0} f(x) = L$ if and only if $\lim_{x\to 0^-} f(x) = L$ and $\lim_{x\to 0^+} f(x) = L$.

We can compute the 1-sided limits.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+2) = \lim_{x\to 0^-} x + \lim_{x\to 0^-} 2$$

$$= 0 + 2 = 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2x-3)$$

$$= \lim_{x\to 0^+} 2x - \lim_{x\to 0^+} 3$$

$$= 2\lim_{x\to 0^+} x - \lim_{x\to 0^+} 3 = 2\cdot 0 - 3 = -3$$

$\lim_{x\to 0^-} f(x) = 2$ and $\lim_{x\to 0^+} f(x) = -3$

$\lim_{x\to 0} f(x)$ DNE

Question

(1) $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ 3 - x, & x > 1 \end{cases}$

(a) 4

(b) 2

(c) 1

(d) DNE

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+1) = 2$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3-x) = 2$

Additional Limit Law Theorems

Suppose $\lim_{x \to c} f(x) = L$ and n is a positive integer.

Theorem: (Power) $\lim_{x \to c} (f(x))^n = L^n$

Note in particular that this tells us that $\lim_{x \to c} x^n = c^n$.

Theorem: (Root) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (if this is defined)

Combining the sum, difference, constant multiple and power laws:
Theorem: If $P(x)$ is a polynomial, then

$$\lim_{x \to c} P(x) = P(c).$$

To take the limit, plug in c.

Question

(1) $\lim_{x \to 2} (3x^2 - 4x + 7) =$

(a) 7

(b) DNE

(c) -11

(d) 11

Let $P(x) = 3x^2 - 4x + 7$

this is a polynomial.

$P(2) = 3 \cdot 2^2 - 4 \cdot 2 + 7$

$= 12 - 8 + 7 = 11$

Notation Reminder

The notation "$\lim_{x \to c}$" is **always** followed by a function expression and never immediately by an equal sign.

Question

(2) Suppose that we have determined that $\lim_{x\to 7} f(x) = 13$.

True or False: It is acceptable to write this as

$$\text{"}\lim_{x\to 7} = 13\text{"}$$

This is similar to writing "$\sqrt{} = 13$"

Additional Limit Law Theorems

$$\text{Suppose} \quad \lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M \quad \text{and} \quad M \neq 0$$

Theorem: (Quotient) $\displaystyle \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

Combined with our result for polynomials:

Theorem: If $R(x) = \frac{p(x)}{q(x)}$ is a rational function, and c is in the domain of R, then

$$\lim_{x\to c} R(x) = R(c).$$

Examples

Evaluate $\lim_{x\to 2} \dfrac{x^2+5}{x^2+x-1}$

$R(x) = \dfrac{x^2+5}{x^2+x-1}$ is a rational function. Is 2 in its domain?

$2^2+2-1 = 4+2-1 = 5 \neq 0$ so yes, 2 is in its domain.

So $\lim_{x\to 2} \dfrac{x^2+5}{x^2+x-1} = \dfrac{2^2+5}{2^2+2-1} = \dfrac{9}{5}$

Examples

Evaluate $\lim_{x\to 1} \frac{\sqrt{x+1}}{x+5}$

Note $\lim_{x\to 1} (x+5) = 1+5 = 6$ $\quad$ $x+5$ is polynomial

and $\lim_{x\to 1} (x+1) = 1+1 = 2$ so $\lim_{x\to 1} \sqrt{x+1} = \sqrt{2}$

So

$$\lim_{x\to 1} \frac{\sqrt{x+1}}{x+5} = \frac{\sqrt{2}}{6}$$

Additional Techniques: When direct laws fail

Evaluate if possible $\displaystyle\lim_{x\to 2}\frac{x^2-x-2}{x^2-4}$

$\dfrac{x^2-x-2}{x^2-4}$ is rational, but 2 is not in its domain.

Note $2^2-2-2=4-2-2=0$

For $p(x)=x^2-x-2$ and $q(x)=x^2-4$, since $p(2)=0$ we know that $x-2$ is a factor of p.

Similarly $q(2)=0$ so $x-2$ is a factor of q.

We can try to cancel this common factor.

$$\lim_{x\to 2} \frac{x^2-x-2}{x^2-4} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x+2)}$$

$$= \lim_{x\to 2} \frac{x+1}{x+2} = \frac{2+1}{2+2} = \frac{3}{4}$$

Additional Techniques: When direct laws fail

Evaluate if possible $\displaystyle\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$

$\displaystyle\lim_{x\to 1}(x-1)=0$ but also $\displaystyle\lim_{x\to 1}\left(\sqrt{x+3}-2\right)=0$

The latter suggests that $x-1$ is a "factor" of $\sqrt{x+3}-2$. We'll use the conjugate of $\sqrt{x+3}-2$, namely $\sqrt{x+3}+2$.

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\right)\cdot\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right)$$

multiply by 1

$$= \lim_{x\to 1} \frac{\left(\sqrt{x+3}\right)^2 - 2\sqrt{x+3} + 2\sqrt{x+3} - 4}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x\to 1} \frac{x+3-4}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}\left(\sqrt{x+3}+2\right)} = \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{1+3}+2}$$

$$= \frac{1}{4}$$

Observations

In limit taking, the form "$\frac{0}{0}$" sometimes appears. This is called an **indeterminate form.** Standard strategies are

(1) Try to factor the numerator and denominator to see if a common factor–$(x - c)$–can be cancelled.

(2) If dealing with roots, try rationalizing to reveal a common factor.

The form

$$\text{"}\frac{\text{nonzero constant}}{0}\text{"}$$

is not indeterminate. It is undefined. When it appears, the limit doesn't exist.