Section 1.8: Limits of Functions Using Numerical and Graphical Techniques

Definition: Let \( f \) be defined on an open interval containing the number \( c \) except possibly at \( c \). Then

\[
\lim_{x \to c} f(x) = L
\]

provided the value of \( f(x) \) can be made arbitrarily close to the number \( L \) by taking \( x \) sufficiently close to \( c \) but not equal to \( c \).
One Sided Limits

**Left Hand Limit:** We write

\[ \lim_{x \to c^-} f(x) = L_L \]

and say the limit as \( x \) approaches \( c \) from the left of \( f(x) \) equals \( L_L \) provided we can make \( f(x) \) arbitrarily close to the number \( L_L \) by taking \( x \) sufficiently close to, but less than \( c \).

**Right Hand Limit:** We write

\[ \lim_{x \to c^+} f(x) = L_R \]

and say the limit as \( x \) approaches \( c \) from the right of \( f(x) \) equals \( L_R \) provided we can make \( f(x) \) arbitrarily close to the number \( L_R \) by taking \( x \) sufficiently close to, but greater than \( c \).
Observations

**Observation 1:** The limit $L$ of a function $f(x)$ as $x$ approaches $c$ does not depend on whether $f(c)$ exists or what it’s value may be.

**Observation 2:** If $\lim_{{x \to c}} f(x) = L$, then the number $L$ is unique. That is, a function can not have two different limits as $x$ approaches a single number $c$.

**Observation 3:** A function need not have a limit as $x$ approaches $c$. If $f(x)$ can not be made arbitrarily close to any one number $L$ as $x$ approaches $c$, then we say that $\lim_{{x \to c}} f(x)$ **does not exist** (shorthand DNE).
A Limit Failing to Exist

Consider \( H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \). Evaluate if possible

\[
\lim_{x \to 0^-} H(x), \quad \lim_{x \to 0^+} H(x), \quad \text{and} \quad \lim_{x \to 0} H(x)
\]

\[\lim_{x \to 0^-} H(x) = 0 \quad \text{from the graph}\]

\[\lim_{x \to 0^+} H(x) = 1 \quad \text{from the graph}\]
\[ \lim_{x \to 0} H(x) \text{ Does not exist (DNE)} \]

There is no one number \( L \) that \( H(x) \) values get arbitrarily close to.

This is called a jump when
\[ \lim_{x \to c^-} f(x) = L_1 \quad \lim_{x \to c^+} f(x) = L_2 \]

and \( L_1 \neq L_2 \).
Weakness of Technology

Suppose we wish to investigate

\[ \lim_{x \to 0} \sin \left( \frac{\pi}{x^2} \right). \]

We consider values of \(x\) closer to zero, and plug them into a calculator. Let’s look at two attempts.

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<tr>
<th>(x)</th>
<th>(\sin \left( \frac{\pi}{x^2} \right))</th>
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</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>0</td>
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<tr>
<td>-0.01</td>
<td>0</td>
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<tr>
<td>-0.001</td>
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<td>-(\frac{2}{3})</td>
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**Limit looks like zero**
Weakness of Technology

In every interval containing zero, the graph of \( \sin(\frac{\pi}{x^2}) \) passes through every \( y \)-value between \(-1\) and \(1\) infinitely many times.

Figure: \( y = \sin \left( \frac{\pi}{x^2} \right) \)
Evaluating Limits

As this example illustrates, we would like to avoid too much reliance on technology for evaluating limits. The next section will be devoted to techniques for doing this for reasonably well behaved functions. We close with one theorem.

**Theorem:** Let \( f \) be defined on an open interval containing \( c \) except possible at \( c \). Then

\[
\lim_{{x \to c}} f(x) = L
\]

if and only if

\[
\lim_{{x \to c^-}} f(x) = L \quad \text{and} \quad \lim_{{x \to c^+}} f(x) = L.
\]
Limits from a graph

Evaluate \( \lim_{{x \to 1^-}} f(x) \)
Limits from a graph

\[ \lim_{{x \to 1^-}} f(x) = 4 \]

\( y = f(x) \)

Points on the graph:
- (-4, 2)
- (1, 4)
- (6, 2)
- (6, 5)
Question

\[
\lim_{x \to 1^+} f(x) = \quad \text{(a) 4} \quad \text{(b) -2} \quad \text{(c) DNE} \quad \text{(d) 1}
\]

aside:
\[
\lim_{x \to 1} f(x) \text{ DNE}
\]

\[
y = f(x)
\]
Section 1.2: Limits of Functions Using Properties of Limits

We begin with two of the simplest limits we may encounter.

**Theorem:** If \( f(x) = A \) where \( A \) is a constant, then for any real number \( c \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} A = A
\]

**Theorem:** If \( f(x) = x \), then for any real number \( c \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} x = c
\]
Examples

(a) \[ \lim_{x \to 0} 7 = 7 \]

Limit of a constant is that constant.

(b) \[ \lim_{x \to \pi^+} 3\pi = 3\pi \]

We know that

\[ \lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L \]

(c) \[ \lim_{x \to -\sqrt{5}} x = -\sqrt{5} \]

Since \[ \lim_{x \to c} x = c \]
Question

\[
\lim_{x \to 4^-} x = \quad \lim_{x \to 4} x = 4
\]

(a) \(x\)

(b) -4

(c) 4

(d) the one sided limit can’t be determined
Additional Limit Law Theorems

Suppose

\[\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M, \quad \text{and } k \text{ is constant.}\]

Theorem: (Sums) \ \ \ \lim_{x \to c} (f(x) + g(x)) = L + M

Theorem: (Differences) \ \ \ \lim_{x \to c} (f(x) - g(x)) = L - M

Theorem: (Constant Multiples) \ \ \ \lim_{x \to c} kf(x) = kL

Theorem: (Products) \ \ \ \lim_{x \to c} f(x)g(x) = LM
Examples

Use the limit law theorems to evaluate if possible

(a) \( \lim_{x \to 2} (3x + 2) = \lim_{x \to 2} 3x + \lim_{x \to 2} 2 \quad \text{(sum)} \)

\[ = 3 \lim_{x \to 2} x + \lim_{x \to 2} 2 \quad \text{(constant multiple)} \]

\[ = 3 \cdot 2 + 2 = 8 \]
Examples
Use the limit law theorems to evaluate if possible

(b) \( \lim_{x \to -3} (x+1)^2 \)

Consider \( \lim_{x \to -3} (x+1) = \lim_{x \to -3} x + \lim_{x \to -3} 1 \)

\[ = -3 + 1 = -2 \]

\( \lim_{x \to -3} (x+1)^2 = \left( \lim_{x \to -3} (x+1) \right)^2 = (-2)^2 = 4 \)
Examples

Use the limit law theorems to evaluate if possible

(c) \( \lim_{x \to 0} f(x) \) where \( f(x) = \begin{cases} 
  x + 2, & x < 0 \\
  1, & x = 0 \\
  2x - 3, & x > 0 
\end{cases} \)

\[
\lim_{x \to 0^-} f(x) = L \text{ if and only if } \lim_{x \to 0^-} f(x) = L \text{ and } \lim_{x \to 0^+} f(x) = L.
\]

we can compute the 1-sided limits.

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+2) = \lim_{x \to 0^-} x + \lim_{x \to 0^-} 2
\]
\[
= 0 + 2 = 2
\]
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x - 3) \\
= \lim_{x \to 0^+} 2x - \lim_{x \to 0^+} 3 \\
= 2 \lim_{x \to 0^+} x - \lim_{x \to 0^+} 3 = 2 \cdot 0 - 3 = -3
\]

\[
\lim_{x \to 0^-} f(x) = 2 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = -3
\]

\[
\lim_{x \to 0} f(x) \text{ DNE}
\]
Question

(1) \( \lim_{x \to 1} f(x) \) where \( f(x) = \begin{cases} 
    x^2 + 1, & x \leq 1 \\
    3 - x, & x > 1 
\end{cases} \)

(a) 4
(b) 2
(c) 1
(d) DNE
Additional Limit Law Theorems

Suppose \( \lim_{x \to c} f(x) = L \) and \( n \) is a positive integer.

**Theorem: (Power)** \( \lim_{x \to c} (f(x))^n = L^n \)

Note in particular that this tells us that \( \lim_{x \to c} x^n = c^n \).

**Theorem: (Root)** \( \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} \) (if this is defined)

Combining the sum, difference, constant multiple and power laws:
**Theorem:** If \( P(x) \) is a polynomial, then
\[
\lim_{x \to c} P(x) = P(c).
\]

To take the limit, plug in \( c \).
(1) \[ \lim_{{x \to 2}} (3x^2 - 4x + 7) = \]

(a) 7

(b) DNE

(c) −11

(d) 11

Let \( P(x) = 3x^2 - 4x + 7 \)

this is a polynomial.

\[ P(2) = 3 \cdot 2^2 - 4 \cdot 2 + 7 = 12 - 8 + 7 = 11 \]
Notation Reminder

The notation \( \lim_{x \to c} \) is always followed by a function expression and never immediately by an equal sign.
(2) Suppose that we have determined that \( \lim_{x \to 7} f(x) = 13 \).

**True or False:** It is acceptable to write this as

\[
\lim_{x \to 7} = 13
\]

This is similar to writing \( \sqrt{13} = 13 \)
Additional Limit Law Theorems

Suppose \( \lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M \) and \( M \neq 0 \)

**Theorem:** (Quotient) \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \)

Combined with our result for polynomials:

**Theorem:** If \( R(x) = \frac{p(x)}{q(x)} \) is a rational function, and \( c \) is in the domain of \( R \), then

\[ \lim_{x \to c} R(x) = R(c). \]
Examples

Evaluate \( \lim_{x \to 2} \frac{x^2 + 5}{x^2 + x - 1} \)

\( R(x) = \frac{x^2 + 5}{x^2 + x - 1} \) is a rational function. Is 2 in its domain?

\[ 2^2 + 2 - 1 = 4 + 2 - 1 = 5 \neq 0 \] so yes, 2 is in its domain.

\[ \therefore \lim_{x \to 2} \frac{x^2 + 5}{x^2 + x - 1} = \frac{2^2 + 5}{2^2 + 2 - 1} = \frac{9}{5} \]
Examples

Evaluate \( \lim_{x \to 1} \frac{\sqrt{x + 1}}{x + 5} \)

\begin{align*}
\text{Note: } & \lim_{x \to 1} (x + 5) = 1 + 5 = 6 \\
\text{and: } & \lim_{x \to 1} (x + 1) = 1 + 1 = 2 \quad \text{so } \lim_{x \to 1} \sqrt{x + 1} = \sqrt{2} \\
\text{so: } & \lim_{x \to 1} \frac{\sqrt{x + 1}}{x + 5} = \frac{\sqrt{2}}{6}
\end{align*}
Additional Techniques: When direct laws fail

Evaluate if possible \( \lim_{{x \to 2}} \frac{x^2 - x - 2}{x^2 - 4} \)

\[
\frac{x^2 - x - 2}{x^2 - 4} \text{ is rational, but 2 is not in its domain.}
\]

Note \( 2^2 - 2 - 2 = 4 - 2 - 2 = 0 \)

For \( p(x) = x^2 - x - 2 \) and \( q(x) = x^2 - 4 \), since \( p(2) = 0 \)

we know that \( x-2 \) is a factor of \( p \).

Similarly \( q(2) = 0 \) so \( x-2 \) is a factor of \( q \).
We can try to cancel this common factor.

\[
\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)(x+1)}{(x-2)(x+2)}
\]

\[
= \lim_{x \to 2} \frac{x+1}{x+2} = \frac{2+1}{2+2} = \frac{3}{4}
\]
Additional Techniques: When direct laws fail

Evaluate if possible \( \lim_{x \to 1} \frac{\sqrt{x + 3} - 2}{x - 1} \)

\[
\lim_{x \to 1} (x - 1) = 0 \quad \text{but also} \quad \lim_{x \to 1} (\sqrt{x + 3} - 2) = 0
\]

The latter suggests that \( x - 1 \) is a "factor" of \( \sqrt{x + 3} - 2 \). We will use the conjugate of \( \sqrt{x + 3} - 2 \), namely \( \sqrt{x + 3} + 2 \).
\[
\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1} = \lim_{x \to 1} \left( \frac{\sqrt{x+3} - 2}{x - 1} \right) \cdot \left( \frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2} \right) \\
\approx \lim_{x \to 1} \frac{(\sqrt{x+3})^2 - 2\sqrt{x+3} + 2\sqrt{x+3} - 4}{(x-1)(\sqrt{x+3} + 2)} \\
= \lim_{x \to 1} \frac{x+3 - 4}{(x-1)(\sqrt{x+3} + 2)} \\
= \lim_{x \to 1} \frac{x-1}{(x-1)(\sqrt{x+3} + 2)} = \lim_{x \to 1} \frac{1}{\sqrt{x+3} + 2} = \frac{1}{4}
\]
Observations

In limit taking, the form \( \frac{0}{0} \) sometimes appears. This is called an **indeterminate form**. Standard strategies are

(1) Try to factor the numerator and denominator to see if a common factor\(-(x - c)\)-can be cancelled.

(2) If dealing with roots, try rationalizing to reveal a common factor.

The form

\[
\frac{\text{nonzero constant}}{0}
\]

is not indeterminate. It is undefined. When it appears, the limit doesn’t exist.