Section 1.7: Limits of Functions Using Numerical and Graphical Techniques

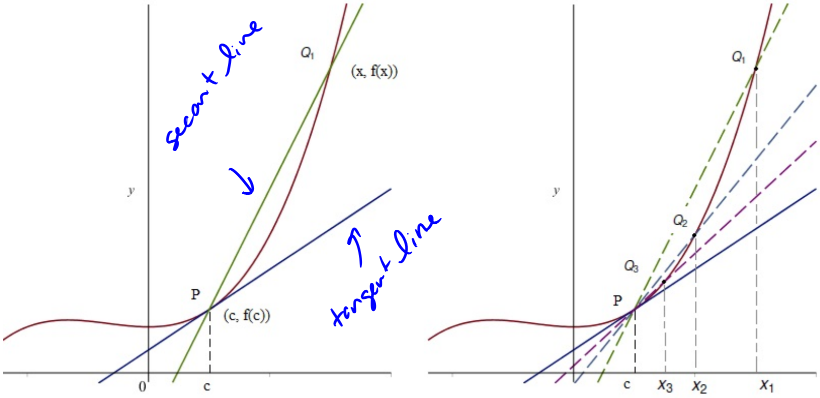
Recall: For a line $y = mx + b$, the slope tells us how a change in x (Δx) causes a change in y (Δy). In fact, the slope $m = \frac{\Delta y}{\Delta x}$.

For a non-line curve, $y = f(x)$ we wanted to define *slope*. We still want slope to say something about how y changes if x changes. But we don't expect slope to be the same number for every x.

We want to define the slope of a **tangent line**.

Slope of the Tangent Line

We consider a sequence of points $Q_1 = (x_1, f(x_1))$, $Q_2 = (x_2, f(x_2))$, and so forth in such a way that the x-values are getting closer to c. Note that the resulting secant lines tend to have slopes closer to that of the tangent line.



Slope of the Tangent Line

$m_{sec} = \frac{f(x) - f(c)}{x - c}$

m_{sec} m_{tan}

This is a limit process. We can write this as

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

(read "the limit as x approaches c of . . . ").

This required us to define what a limit is.

A Working Definition of a Limit

Definition: Let f be defined on an open interval containing the number c except possibly at c. Then

$$\lim_{x \to c} f(x) = L$$

provided the value of $f(x)$ can be made arbitrarily close to the number L by taking x sufficiently close to c but not equal to c.

We considered the example (using a calculator)

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

$\lim_{x \to c} f(x) = L$

here $c = 0$ and $L = 1$

x	$f(x) = \frac{e^x - 1}{x}$
-0.1	0.9516
-0.01	0.9950
-0.001	0.9995
0	undefined
0.001	1.0005
0.01	1.0050
0.1	1.0517

x close to c but less than c

x close to c but greater than c

From the values, we conclude that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

Left and Right Hand Limits

In our examples, we considered x-values to the left (less than) and to the right (greater than) c. This illustrates the notion of **one sided limits**. We have a special notation for this.

Left Hand Limit: We write

$$\lim_{x \to c^-} f(x) = L_L$$

and say *the limit as* x *approaches* c *from the left of* $f(x)$ *equals* L_L *provided we can make* $f(x)$ *arbitrarily close to the number* L_L *by taking* x *sufficiently close to, but less than* c*.*

Left and Right Hand Limits

Right Hand Limit: We write

$$\lim_{x \to c^+} f(x) = L_R$$

and say *the limit as x approaches c from the right of $f(x)$ equals L_R provided we can make $f(x)$ arbitrarily close to the number L_R by taking x sufficiently close to, but greater than c.*

Some other common phrases:

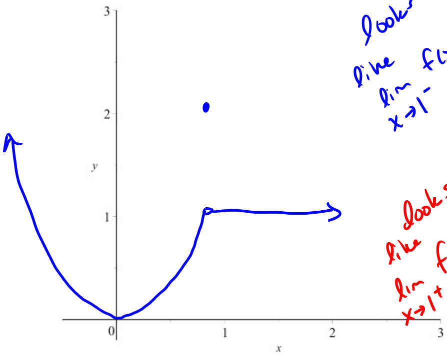
"from the left" is the same as "from below"

"from the right" is the same as "from above."

Example

Plot the function $f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1 \\ 1, & x > 1 \end{cases}$ Investigate $\lim\limits_{x \to 1} f(x)$ using the graph.

$c=1$



looks like $\lim\limits_{x \to 1^-} f(x) = 1$

looks like $\lim\limits_{x \to 1^+} f(x) = 1$

x	$f(x)$
0.9	$(0.9)^2 = 0.81$
0.99	$(0.99)^2 = 0.9801$
0.999	$(0.999)^2 = 0.998001$
1	2
1.001	1
1.01	1
1.1	1

In fact $\lim\limits_{x \to 1} f(x) = 1$

Observations

Observation 1: The limit L of a function $f(x)$ as x approaches c does not depend on whether $f(c)$ exists or what it's value may be.

Observation 2: If $\lim_{x \to c} f(x) = L$, then the number L is unique. That is, a function can not have two different limits as x approaches a single number c.

Observation 3: A function need not have a limit as x approaches c. If $f(x)$ can not be made arbitrarily close to any one number L as x approaches c, then we say that $\lim_{x \to c} f(x)$ **does not exist** (shorthand **DNE**).

Questions

(1) **True or False** It is possible that both $\lim_{x \to 3} f(x) = 5$ AND $f(3) = 7$.

f(c) doesn't have to affect the limit.

(2) **True or False** It is possible that both $\lim_{x \to 3} f(x) = 5$ AND $\lim_{x \to 3} f(x) = 7$.

Limits are unique.