Section 1.8: Limits of Functions Using Properties of Limits

We recall **Theorem:** If $f(x) = A$ where A is a constant, then for any real number c

$$\lim_{x \to c} f(x) = \lim_{x \to c} A = A$$

Theorem: If $f(x) = x$, then for any real number c

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

Limit Law Theorems

Suppose

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M, \quad \text{and } k \text{ is constant.}$$

Theorem: (Sums) $\lim_{x\to c}(f(x)+g(x)) = L+M$

Theorem: (Differences) $\lim_{x\to c}(f(x)-g(x)) = L-M$

Theorem: (Constant Multiples) $\lim_{x\to c} kf(x) = kL$

Theorem: (Products) $\lim_{x\to c} f(x)g(x) = LM$

Theorem: (Quotient) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad \text{if } M \neq 0$

Limit Law Theorems

Theorem: (Power) $\lim_{x \to c} (f(x))^n = L^n$

Note in particular that this tells us that $\lim_{x \to c} x^n = c^n$.

Theorem: (Root) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (if this is defined)

Theorem: If $R(x)$ is a rational function, and c is in the domain of R, then

$$\lim_{x \to c} R(x) = R(c).$$

Note that this includes all polynomials, and recall that the domain of any polynomial is all reals.

Observations

In limit taking, the form "$\frac{0}{0}$" sometimes appears. This is called an **indeterminate form.** Standard strategies are

(1) Try to factor the numerator and denominator to see if a common factor–$(x - c)$–can be cancelled.

(2) If dealing with roots, try rationalizing to reveal a common factor.

The form

$$\text{"}\frac{\text{nonzero constant}}{0}\text{"}$$

is not indeterminate. It is undefined. When it appears, the limit doesn't exist.

Example

Evaluate if possible $\lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$.

Top and bottom both go to zero as $x \to 2$.

we'll rationalize.

$$\lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}} = \lim_{x\to 2} \left(\frac{x-2}{\sqrt{x}-\sqrt{2}}\right)\cdot\left(\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}\right)$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})}{\cancel{x-2}}$$

$$= \lim_{x\to 2} (\sqrt{x}+\sqrt{2}) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$$

Question

$$\lim_{x\to 1} \frac{x^2-2x+1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x-1)}{\cancel{(x-1)}(x+1)}$$

$$= \lim_{x\to 1} \frac{x-1}{x+1} = \frac{1-1}{1+1} = \frac{0}{2} = 0$$

(a) DNE

(b) 1

(c) may exist, but can't be determined without a graph

(d) 0

Example

Let $f(x) = x^3 + 2x$. Determine the difference quotient

$$\frac{f(x+h) - f(x)}{h} \quad \text{for} \quad h \neq 0.$$

Next, take the limit as $h \to 0$ of this difference quotient.

$$f(x) = x^3 + 2x$$

$$f(x+h) = (x+h)^3 + 2(x+h) = (x+h)(x+h)^2 + 2(x+h)$$

$$= x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h$$

parenthesis required

$$\frac{f(x+h) - f(x)}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - (x^3 + 2x)}{h}$$

$$= \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{2x} + 2h - \cancel{x^3} - \cancel{2x}}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 + 2h}{h}$$

$$= \frac{\cancel{h}(3x^2 + 3xh + h^2 + 2)}{\cancel{h}}$$

$$= 3x^2 + 3xh + h^2 + 2$$

We can take the limit now as $h \to 0$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \left(3x^2 + 3xh + h^2 + 2\right)$$

$$= 3x^2 + 3x \cdot 0 + 0^2 + 2$$

$$= 3x^2 + 2$$

Since x doesn't depend on h, we treat x just like a constant when taking this limit.

Section 1.3: Continuity

We have seen that their may or may not be a relationship between the quantities

$$\lim_{x \to c} f(x) \quad \text{and} \quad f(c).$$

One or the other (or both) may fail to exist. And even if both exist, they need not be equivalent.

We've also seen that for polynomials at least, that the limit at a point is the same as the function value at that point. Here, we explore this property that polynomials (and lots of other functions, but not all) share.

Definition: Continuity at a Point

Definition: A function f is continuous at a number c if

$$\lim_{x \to c} f(x) = f(c).$$

Note that three properties are contained in this statement:
(1) $f(c)$ is defined (i.e. c is in the domain of f),

(2) $\lim_{x \to c} f(x)$ exists, and

(3) the limit actually equals the function value.

If a function f is not continuous at c, we may say that f is **discontinuous** at c

Polynomials and Rational Functions

In the previous section, we saw that:

If P is any polynomial and c is any real number, then $\lim_{x \to c} P(x) = P(c)$, and

If R is any rational function and c is any number in the domain of R, then $\lim_{x \to c} R(x) = R(c)$.

Conclusion Theorem: Every rational function[1] is continuous at each number in its domain.

[1] Note that polynomials can be lumped in to the set of all rational functions.

Examples: Determine where each function is discontinuous.

(a) $g(t) = \dfrac{t^2 - 9}{t + 3}$

g is rational, so it's continuous on its domain. So the question becomes, "what is not in the domain?"

c is not in the domain if the denominator is zero there.

$t + 3 = 0 \Rightarrow t = -3.$

g is discontinuous @ -3 because $g(-3)$ isn't defined.

As an aside, note that

$$\lim_{t\to -3} g(t) = \lim_{t\to -3} \frac{t^2-9}{t+3} = \lim_{t\to -3} \frac{\cancel{(t+3)}(t-3)}{\cancel{t+3}}$$

$$= \lim_{t\to -3} (t-3) = -3-3 = -6$$

This doesn't change the fact that g is discontinuous @ -3.

(b) $f(x) = \begin{cases} 2x, & x < 1 \\ x^2 + 1, & 1 \le x < 2 \\ 3, & x \ge 2 \end{cases}$

Note that the pieces are polynomials.

On $(-\infty, 1)$, $f(x) = 2x$, so it's continuous at each c in $(-\infty, 1)$

On $(1, 2)$, $f(x) = x^2 + 1$, so it's continuous at each c in $(1, 2)$

On $(2, \infty)$, $f(x) = 3$, so it's continuous @ each c in $(2, \infty)$.

We'll check what happens @ $x = 1$ and @ $x = 2$.

Check @ 1:

① Is $f(1)$ defined? yes, $f(1) = 1^2+1 = 2$

② Does $\lim_{x\to 1} f(x)$ exist?

$$\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 2x = 2\cdot 1 = 2 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (x^2+1) = 1^2+1 = 2 \end{aligned}\right\} \text{ yes, } \lim_{x\to 1} f(x) = 2$$

③ Is $\lim_{x\to 1} f(x) = f(1)$? Yes, they're both 2.

f is continuous at 1.

Check for continuity @ 2:

① Is $f(2)$ defined? yes, $f(2)=3$

② Does $\lim_{x\to 2} f(x)$ exist?

$$\left.\begin{aligned} &\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2+1) = 2^2+1 = 5 \\ &\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3 = 3 \end{aligned}\right\} \text{No, } \lim_{x\to 2} f(x) \text{ DNE}$$

So f is discontinuous at 2 since the limit doesn't exist.

Question

Determine whether f is continuous at 1 where $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

$f(1) = 2$

(a) No because $f(1)$ is not defined.

(b) Yes because all three conditions hold.

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1}$

(c) No because $\lim_{x \to 1} f(x)$ doesn't exist.

$= 2$

(d) No because f is piecewise defined.

Removable and Jump Discontinuities

Definition: Let f be defined on an open interval containing c except possibly at c. If $\lim_{x \to c} f(x)$ exists, but f is discontinuous at c, then f has a **removable discontinuity** at c.

The function can be redefined at the point c so that the result is continuous.

Definition: If $\lim_{x \to c^-} f(x) = L_1$ and $\lim_{x \to c^+} f(x) = L_2$ where $L_1 \neq L_2$ (i.e. both one sided limits exist but are different), then f has a **jump discontinuity** at c.

Removable and Jump Discontinuities

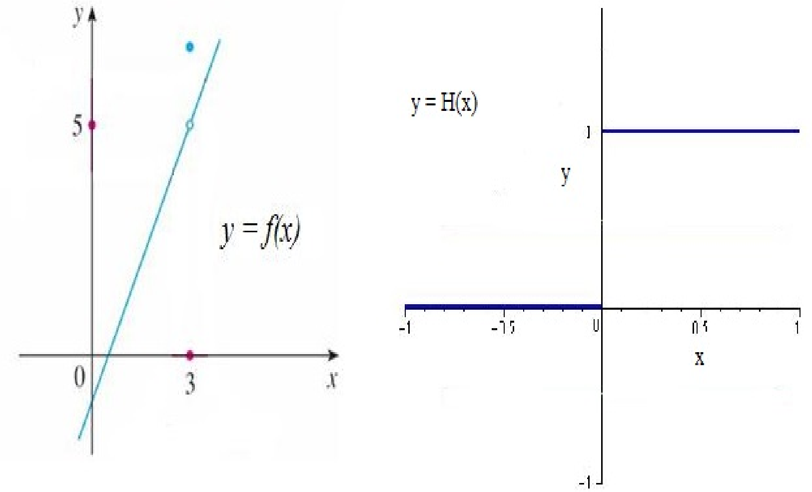


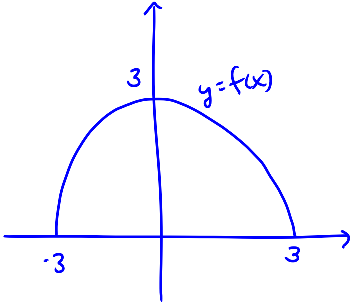
Figure: Example of a removable (left) discontinuity and a jump (right) discontinuity.

One Sided Continuity Example:

Consider the function $f(x) = \sqrt{9 - x^2}$. Plot a rough sketch of the graph of f, and determine its domain.

Let $y = \sqrt{9 - x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$

Top half of a circle



From the graph, the domain is $[-3, 3]$.

Alternatively, we need $9 - x^2 \geq 0$

$$\Rightarrow \quad 9 \geq x^2$$

$$\Rightarrow \quad \sqrt{9} \geq \sqrt{x^2} \quad \Rightarrow \quad 3 \geq |x|$$

i.e. $\quad -3 \leq x \leq 3$

we find the same domain.

$f(x) = \sqrt{9 - x^2}$

Note that f is continuous on $-3 < x < 3$. What can be said about

$$\lim_{x \to -3} f(x) \qquad \text{or} \qquad \lim_{x \to 3} f(x)?$$

The limits don't exist because f is not defined on an open interval containing either 3 or -3.

Continuity From the Left & Right

Definition: Let a function f be defined on an interval $[c, b)$. Then f is continuous from the right at c if

$$\lim_{x \to c^+} f(x) = f(c).$$

Let f be defined on an interval $(a, c]$. Then f is continuous from the left at c if

$$\lim_{x \to c^-} f(x) = f(c).$$

Example: $f(x) = \sqrt{9 - x^2}$

Compare $f(-3)$ and $\lim_{x\to -3^+} f(x)$. Is f continuous from the right at -3?

$$f(-3) = \sqrt{9-(-3)^2} = \sqrt{9-9} = \sqrt{0} = 0$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9-x^2} = \sqrt{9-(-3)^2} = 0$$

So f is continuous from the right

@ -3.