Section 1.8: Continuity

Definition: Continuity at a Point A function f is continuous at a number c if

$$\lim_{x \to c} f(x) = f(c).$$

This definition is equivalent to the three statements
(1) $f(c)$ is defined (i.e. c is in the domain of f),

(2) $\lim_{x \to c} f(x)$ exists, and

(3) the limit actually equals the function value.

If a function f is not continuous at c, we may say that f is **discontinuous** at c

Question

Suppose f is continuous at -4 and $f(-4) = 2\pi$. Then

$$\lim_{x \to -4} f(x) = f(-4) = 2\pi$$

(a) -4

(b) -8π

(c) 2π

(d) can't be determined without more information

A Theorem on Continuous Functions

Theorem If f and g are continuous at c and for any constant k, the following are also continuous at c:

$$(i)\, f+g, \quad (ii)\, f-g, \quad (iii)\, kf, \quad (iv)\, fg, \quad \text{and} \quad (v)\, \frac{f}{g}, \text{ if } g(c)\neq 0.$$

In other words, if we combine continuous functions using addition, subtraction, multiplication, division, and using constant factors, the result is also continuous—provided of course that we don't introduce division by zero.

Questions

(1) **True or False** If f is continuous at 3 and g is continuous at 3, then it must be that

$$\lim_{x\to 3} f(x)g(x) = f(3)g(3).$$

By continuity

(2) **True or False** If $f(2) = 1$ and $g(2) = 7$, then it must be that

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{1}{7}.$$

Consider

$$f(x) = \begin{cases} 1, & x \geq 2 \\ 0, & x < 2 \end{cases}$$

$$g(x) = 7$$

Continuity on an Interval

Definition A function is continuous on an interval (a, b) if it is continuous at each point in (a, b). A function is continuous on an interval such as $(a, b]$ or $[a, b)$ or $[a, b]$ provided it is continuous on (a, b) and has one sided continuity at each included end point.

Graphically speaking, if $f(x)$ is continuous on an interval (a, b), then the curve $y = f(x)$ will have no holes or gaps.

Find all values of A such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} x + A, & x < 2 \\ Ax^2 - 3, & 2 \leq x \end{cases}$$

The pieces are continuous.

we need to know what happens @ 2.

We require $\lim_{x \to 2} f(x) = f(2)$.

$f(2) = A(2^2) - 3 = 4A - 3$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+A) = 2+A$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (Ax^2-3) = 4A-3$$

For existence of the limit, it must be that

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$2+A = 4A-3$$

$$5 = 3A \implies A = \frac{5}{3}$$

So f is continuous @ 2 if $A = \frac{5}{3}$.

Then $\lim_{x\to 2} f(x) = 2 + \frac{5}{3} = \frac{11}{3}$

and $f(2) = 4\left(\frac{5}{3}\right) - 3 = \frac{20}{3} - \frac{9}{3} = \frac{11}{3}$

$$f(x) = \begin{cases} x + A, & x < 2 \\ Ax^2 - 3, & 2 \leq x \end{cases}$$

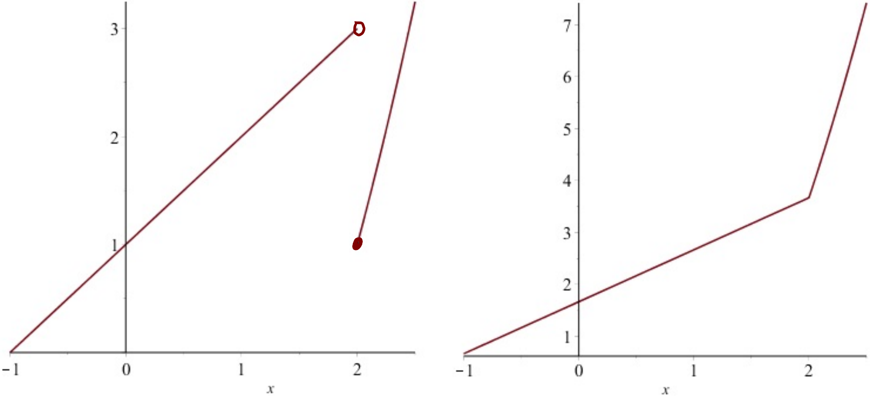

Figure: On the left, $A \neq \frac{5}{3}$; on the right $A = \frac{5}{3}$.

Compositions

Suppose $\lim_{x\to c} g(x) = L$, and f is continuous at L, then

$$\lim_{x\to c} f(g(x)) = f(L) \quad \text{i.e.} \quad \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right).$$

Theorem: If g is continuous at c and f is continuous at $g(c)$, then $(f \circ g)(x)$ is continuous at c.

Essentially, this says that "compositions of continuous functions are continuous."

Example

Suppose we know that $f(x) = e^x$ is continuous on $(-\infty, \infty)$[1]. Evaluate

$$\lim_{x\to\sqrt{\ln(3)}} e^{x^2+\ln(2)}$$

If $g(x) = x^2 + \ln 2$, it's a quadratic and is continuous everywhere.

$f(x)$ is continuous, so $f(g(x))$ is continuous.

$$\lim_{x\to\sqrt{\ln 3}} e^{x^2+\ln 2} = e^{(\sqrt{\ln 3})^2+\ln 2} = e^{\ln 3+\ln 2}$$

$$= e^{\ln 3}\cdot e^{\ln 2} = 3\cdot 2 = 6$$

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[1] This is true.

Inverse Functions

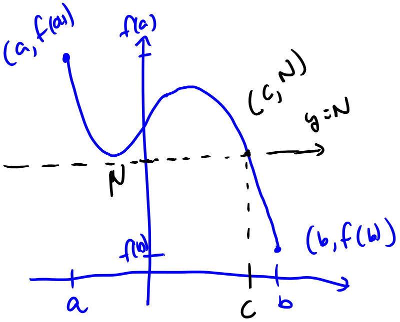
Theorem: If f is a one to one function that is continuous on its domain, then its inverse function f^{-1} is continuous on its domain.

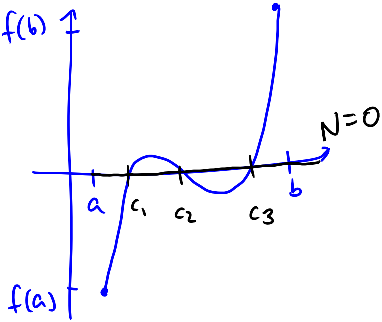
Continuous functions (with inverses) have continuous inverses.

Theorem:

Intermediate Value Theorem (IVT) Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists c in the interval (a, b) such that $f(c) = N$.

A continuous function must take on every value between $f(a)$ and $f(b)$.





Application of the IVT

Show that the equation has at least one solution in the interval.

$$x^3 + x^2 - 4 = 0 \quad 1 \le x \le 2$$

Let $f(x) = x^3 + x^2 - 4$. A solution to the equation would be a root of f. As a polynomial f is continuous at all reals, so it's continuous on $[1, 2]$.

$$f(1) = 1^3 + 1^2 - 4 = 2 - 4 = -2$$

$$f(2) = 2^3 + 2^2 - 4 = 8 + 4 - 4 = 8$$

Since $-2 < 0 < 8$ i.e. 0 is a number between $f(1)$ and $f(2)$, the IVT says there exists some number c in $(1,2)$ such that $f(c) = 0$.

This c is a solution since

$$0 = f(c) = c^3 + c^2 - 4$$

Section 1.4: Limits and Continuity of Trigonometric, Exponential and Logarithmic Functions

Here we list without proof[2] the continuity properties of several well known functions.

$\sin x$: The sine function $y = \sin x$ is continuous on its domain $(-\infty, \infty)$.

$\cos x$: The cosine function $y = \cos x$ is continuous on its domain $(-\infty, \infty)$.

e^x: The exponential function $y = e^x$ is continuous on its domain $(-\infty, \infty)$.

$\ln(x)$: The natural log function $y = \ln(x)$ is continuous on its domain $(0, \infty)$.

[2] You are already familiar with their graphs.

Additional Functions

- By the quotient property, each of $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are continuous on each of their respective domains.

- For $a > 0$ with $a \neq 1$, the function
$$a^x = e^{x \ln a}.$$
By the composition property, each exponential function $y = a^x$ is continuous on $(-\infty, \infty)$.

- For $a > 0$ with $a \neq 1$, the function
$$\log_a(x) = \frac{\ln x}{\ln a}.$$
By the constant multiple property, each logarithm function $y = \log_a(x)$ is continuous on $(0, \infty)$.

What does all this mean?

The common functions we use, polynomial and rational functions, trigonometric functions, and logs and exponentials are continuous **everywhere on their respective domains.**

So, if f is anyone of these functions and c is a number in its domain, then $\lim_{x \to c} f(x) = f(c)$.

Example

Evaluate each limit.

(a) $\lim_{x \to \pi} \cos(x + \sin x)$

$y = x$, $y = \sin x$, $y = \cos y$

are all continuous everywhere.

So $\cos(x + \sin x)$ is continuous @ π

$= \cos(\pi + \sin \pi)$

$= \cos(\pi + 0)$

$= \cos(\pi) = -1$

Example

$\frac{\pi}{4}$ is in the domain of

$y = \tan t$

(b) $\lim_{t \to \frac{\pi}{4}} e^{\tan t}$

$= e^{\tan\frac{\pi}{4}} = e^{1} = e$

Question

Evaluate the limit $\lim_{x\to\pi} \ln(\cos^2 x)$.

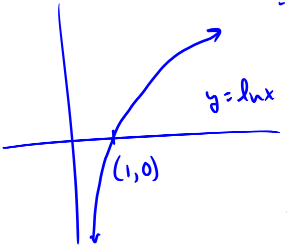
(a) e

(b) 1

(c) DNE

(d) 0

$$\lim_{x \to \pi} \ln(\cos^2 x) = \ln(\cos^2 \pi)$$
$$= \ln((-1)^2)$$
$$= \ln 1 = 0$$



Squeeze Theorem:

Theorem: Suppose $f(x) \leq g(x) \leq h(x)$ for all x in an interval containing c except possibly at c. If

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} g(x) = L.$$

Squeeze Theorem:

$f(x) \le g(x) \le h(x)$

$\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$

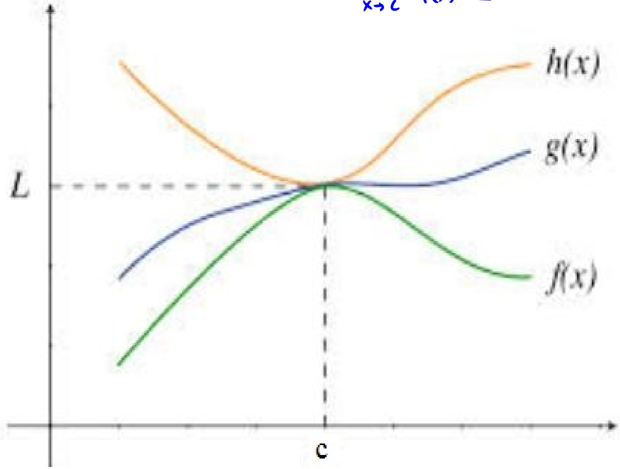


Figure: Graphical Representation of the Squeeze Theorem.

Example: Evaluate

$\sin\frac{1}{\theta}$ isn't defined @ $\theta=0$

But the sine is bounded.

$$\lim_{\theta\to 0}\theta^2\sin\frac{1}{\theta}$$

$$-1 \le \sin\frac{1}{\theta} \le 1$$

Mult. by θ^2
which is positive
for all $\theta \neq 0$

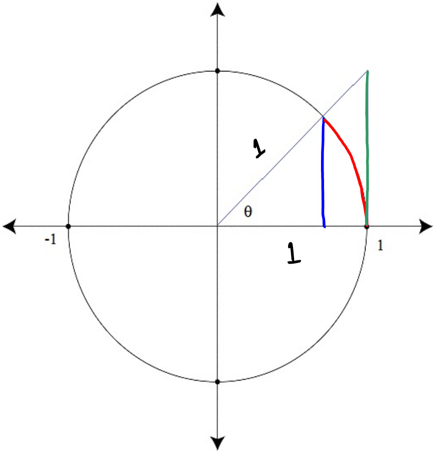
$$-1\cdot\theta^2 \le \theta^2\sin\frac{1}{\theta} \le 1\cdot\theta^2$$

$$\underbrace{-\theta^2}_{f} \le \underbrace{\theta^2\sin\frac{1}{\theta}}_{g} \le \underbrace{\theta^2}_{h}$$

$\lim_{\theta\to 0} -\theta^2 = 0$ and $\lim_{\theta\to 0}\theta^2 = 0$ $\Rightarrow$ $\lim_{\theta\to 0}\theta^2\sin\frac{1}{\theta} = 0$ by the squeeze theorem

A Couple of Important Limits

Theorem: $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta} = 0$



- $\sin\theta$
- θ
- $\tan\theta$

Note $\sin\theta \le \theta \le \tan\theta$

Inequalities

① $\sin\theta \le \theta$ and

② $\theta \le \tan\theta$

① $\sin\theta \le \theta$ for $\theta > 0$ $\quad \sin\theta \le \theta \Rightarrow \dfrac{\sin\theta}{\theta} \le 1$

Note that $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{-\sin\theta}{-\theta} = \dfrac{\sin\theta}{\theta}$

So for all θ close to zero

$$\boxed{\frac{\sin\theta}{\theta} \le 1}$$

② $\theta \le \tan\theta$ for $\theta > 0$ $\quad \theta \le \dfrac{\sin\theta}{\cos\theta}$

Mult. by $\cos\theta$ and divide by θ

$$\cos\theta \le \frac{\sin\theta}{\theta}$$

Since $\cos(-\theta) = \cos\theta$ and $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$

So for all θ close to zero

$$\boxed{\cos\theta \le \frac{\sin\theta}{\theta}}$$

So for $\theta \approx 0$ $\quad \cos\theta \le \frac{\sin\theta}{\theta} \le 1$

$\lim_{\theta\to 0} \cos\theta = \cos 0 = 1$ and $\lim_{\theta\to 0} 1 = 1$

$\Rightarrow$ $\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \text{ by the squeeze thm.}}$

Notational Note

The name of the variable used is irrelevant. That is

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,\quad \lim_{x\to 0}\frac{\sin x}{x}=1,\quad \lim_{\heartsuit\to 0}\frac{\sin\heartsuit}{\heartsuit}=1$$

In fact, this only requires the argument of the sine to match the denominator (exactly) and that this term is tending to zero. For example,

$$\lim_{\theta\to 0}\frac{\sin(6\theta)}{6\theta}=1,\quad \text{and}\quad \lim_{\heartsuit\to 0}\frac{\sin(\pi\heartsuit)}{\pi\heartsuit}=1$$

Examples

Use $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ to evaluate each limit.

If we had $\lim_{x\to 0} \frac{\sin(4x)}{4x}$ the limit would be 1.

(a) $\displaystyle \lim_{x \to 0} \frac{\sin(4x)}{x}$

mult. by 1

$$\lim_{x\to 0} \frac{\sin(4x)}{x} = \lim_{x\to 0} \frac{\sin(4x)}{x} \cdot \left(\frac{4}{4}\right)$$

$$= \lim_{x\to 0} \frac{\sin(4x)}{4x} \cdot \frac{4}{1} = \lim_{x\to 0} 4\left(\frac{\sin(4x)}{4x}\right)$$

$$= 4 \cdot 1 = 4$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

(b) $\displaystyle\lim_{t\to 0} \sin t \cdot \csc(3t)$

$$=\lim_{t\to 0}\frac{\sin t}{\sin(3t)} = \lim_{t\to 0}\frac{\sin t}{1}\;\frac{1}{\sin(3t)}$$

$$=\lim_{t\to 0}\frac{\sin t}{t}\;\frac{t}{\sin(3t)}$$

$$=\lim_{t\to 0}\frac{\sin t}{t}\;\frac{t}{\sin(3t)}\cdot\frac{3}{3}$$

$$=\lim_{t\to 0}\frac{\sin t}{t}\;\frac{3t}{\sin(3t)}\cdot\frac{1}{3}$$

$$= \lim_{t\to 0} \frac{\sin t}{t} \left(\frac{\sin(3t)}{3t}\right)^{-1} \cdot \frac{1}{3}$$

$$= 1 \cdot 1^{-1} \cdot \frac{1}{3} = 1 \cdot 1 \cdot \frac{1}{3} = \frac{1}{3}$$

Questions

(1) Evaluate if possible $\lim_{x \to 0} \frac{\tan(2x)}{4x}$

(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) 2

(d) DNE

$\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$

So

$\frac{\tan(2x)}{4x} = \frac{1}{2\cos(2x)} \cdot \frac{\sin(2x)}{2x}$

A couple of important observations

$$\lim_{x\to 0} \cos x = 1, \quad \text{so for example} \quad \lim_{x\to 0} \frac{\cos x}{x} \text{ DNE}$$

While it is true that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the statement

$$\frac{\sin x}{x} = 1$$

is **always false!** Don't be tempted to write this.

Also remember that $\sin(kx) \neq k sin(x)$. Don't be tempted to try to *factor* out of a trig function.