# Math 5510/Math 4510 - Partial Differential Equations

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Spring 2021

# Outline

# Second Order Differential Equation

Consider the **initial value problem (IVP)**:

$$y'' - y = 0, \qquad y(0) = y_0, \quad \text{and} \quad y'(0) = yp_0.$$

This is a **second order linear homogeneous differential equation**.

Solve this by attempting the solution $y(t) = ce^{\lambda t}$, which results in

$$c\lambda^2 e^{\lambda t} - ce^{\lambda t} = ce^{\lambda t}(\lambda^2 - 1) = 0.$$

This results in the ***characteristic equation***

$$\lambda^2 - 1 = (\lambda + 1)(\lambda - 1) = 0, \qquad \text{so} \qquad \lambda = \pm 1,$$

which gives the general solution:

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

# Second Order Differential Equation

The initial value problem (IVP):

$$y'' - y = 0, \qquad y(0) = y_0, \quad \text{and} \quad y'(0) = yp_0.$$

has the solution

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

From the initial conditions,

$$\begin{aligned} c_1 + c_2 &= y_0, \\ c_1 - c_2 &= yp_0, \end{aligned}$$

which has the unique solution $c_1 = \frac{y_0 + yp_0}{2}$ and $c_2 = \frac{y_0 - yp_0}{2}$.

Thus,

$$y(t) = \frac{y_0 + yp_0}{2} e^t + \frac{y_0 - yp_0}{2} e^{-t} = y_0 \cosh(t) + yp_0 \sinh(t).$$

# First Order System of DEs

Consider the ODE

$$y'' - y = 0.$$

Let $y_1(t) = y(t)$ and $y_2(t) = y'(t) = y_1'(t)$, so $y_2'(t) = y''(t) = y_1(t)$.

The ***second order DE*** can be written as the ***first order system*** of ODEs:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

The ***characteristic equation*** of the matrix satisfies

$$\det |A - \lambda I| = \det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0,$$

which is the same as for the ODE before.

Once again the associated eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$

# First Order System of DEs

Consider the eigenvalue $\lambda_1 = 1$ for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The associated eigenvector is easily seen to be $\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Similarly associated eigenvector for $\lambda_2 = -1$ is $\xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

It follows that the solution to the system of DEs

$$\dot{\mathbf{y}} = A\mathbf{y},$$

is

$$\mathbf{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

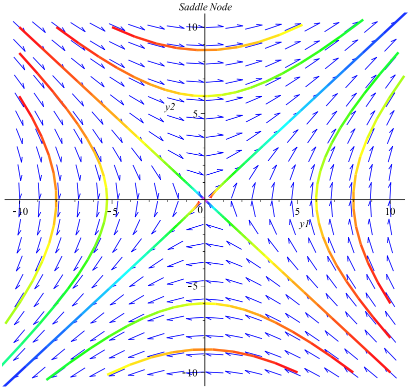
# Phase Portrait

The results above give the general solution

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

This is a **saddle node**.

Solutions move toward the origin in the direction $\xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and move away from origin in the direction $\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for larger $t$



# Boundary Value Problem

Consider the **boundary value problem (BVP)**:

$$y'' - y = 0, \qquad y(0) = A, \quad \text{and} \quad y(1) = B,$$

which again has the general solution $y(t) = c_1 e^t + c_2 e^{-t}$.

With algebra, the **unique solution** becomes

$$y(t) = -\frac{(A\mathrm{e} - B)\,\mathrm{e}^{-t}}{\mathrm{e}^{-1} - \mathrm{e}} + \frac{\left(A\mathrm{e}^{-1} - B\right)\mathrm{e}^{t}}{\mathrm{e}^{-1} - \mathrm{e}}$$

Since $\sinh(t)$ and $\sinh(1-t)$ are linearly independent combinations of $e^t$ and $e^{-t}$, we could write

$$y(t) = d_1 \sinh(t) + d_2 \sinh(1-t).$$

The algebra makes it much easier to see that

$$y(t) = \frac{B}{\sinh(1)} \sinh(t) + \frac{A}{\sinh(1)} \sinh(1-t).$$

# Linear Independence

Below is the definition of **Linear Independence**.

**Definition (Linear Independence)**

Let $V$ be the ***vector space*** of all real valued functions of a real variable $x$. A set of functions, $\{f_i(x)\}_{i=1}^n$, is ***linearly independent*** if and only if a linear combination of those functions,

$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0, \quad \text{for all} \quad x,$$

implies that all the constants, $c_i = 0$.

Consider the set of functions, $\{e^t, e^{-t}\}$ and assume that

$$c_1 e^t + c_2 e^{-t} = 0, \quad \text{for all} \quad t.$$

Solving this equation gives $c_1 e^{2t} = -c_2$, for all $t$, which only occurs when $c_1 = 0$. It follows that $c_2$ is also zero.

# Existence and Uniqueness

Below is an important theorem about the **initial value problem**:

$$y' = f(t,y), \qquad \text{with} \qquad y(0) = 0 \tag{1}$$

**Theorem (Existence and Uniqueness)**

*If $f$ and $\partial f/\partial y$ are continuous in a rectangle $R: |t| \le a, |y| \le b$, then there is some interval $|t| \le h \le |a|$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem (1).*

This theorem states that assuming the function $f$ is smooth, then the ***first order differential equation*** has a ***unique solution*** through a specific ***initial condition***.

Since we are primarily considering $f(t,y)$ ***linear*** in $y$, this theorem is satisfied.

**Does this theorem hold for boundary value problems?**

# Harmonic Oscillator

**Example (Harmonic Oscillator):** Consider the IVP:

$$y'' + y = 0, \qquad y(0) = A, \quad y'(0) = B$$

The ***characteristic equation*** for this ODE is $\lambda^2 + 1 = 0$, which has solutions $\lambda = \pm i$

It follows that the general solution is

$$y(t) = c_1 \cos(t) + c_2 \sin(t).$$

The initial conditions are easily solved to give the ***unique solution***

$$y(t) = A\cos(t) + B\sin(t),$$

which is the classic **harmonic undamped oscillator**.

# Harmonic Oscillator

**Example (Harmonic Oscillator):** Now consider the BVP:

$$y'' + y = 0, \qquad y(0) = A, \quad y(1) = B,$$

which again has the general solution

$$y(t) = c_1 \cos(t) + c_2 \sin(t).$$

The boundary conditions are easily solved to give

$$y(t) = A\cos(t) + \frac{B - A\cos(1)}{\sin(1)} \sin(t).$$

This again gives a ***unique solution***, but the denominator of $\sin(1)$ suggests potential problems at certain $t$ values.

# Harmonic Oscillator

**Example (Harmonic Oscillator):** Now consider the BVP:

$$y'' + y = 0, \qquad y(0) = A, \quad y(\pi) = B,$$

which again has the general solution

$$y(t) = c_1 \cos(t) + c_2 \sin(t).$$

The condition $y(0) = A$ implies $c_1 = A$. However, $y(\pi) = B$ gives

$$y(\pi) = A\cos(\pi) + c_2 \sin(\pi) = -A = B.$$

This only has a solution if $B = -A$. Furthermore, if $B = -A$, the arbitrary constant $c_2$ remains undetermined, so takes any value.

- If $B \neq -A$, then **no solution exists**.
- If $B = -A$, then **infinity many solutions exist** and satisfy

$$y(t) = A\cos(t) + c_2 \sin(t), \quad \text{where } c_2 \text{ is arbitrary.}$$

# General Case

**Theorem (Boundary Value Problem)**

*Consider the second order linear BVP*

$$y'' + py' + qy = 0, \qquad y(a) = A, \quad y(b) = B,$$

*where $p$, $q$, $a \neq b$, $A$, and $B$ are constants. Exactly one of the following conditions hold:*

- *There is a **unique solution** to the BVP.*
- *There is **no solution** to the BVP.*
- *There are **infinity many solutions** to the BVP.*

The previous example demonstrates this theorem well, and this theorem will be critical to solving many of our PDEs this semester.