Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition: Linear Dependence/Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \ldots, c_p$ *at least one of which is nonzero* such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

(i.e. Provided the homogeneous equation posses a nontrivial solution.)

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Theorem on Linear Independence

Theorem: The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Example

Determine if the set is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

One approach is to set $A = [\vec{v}_1 \; \vec{v}_2]$ and consider $A\vec{x} = \vec{0}$.

Take the augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 = 0$
(no free variables)

The vectors are linearly independent.

Example

Determine if the set is linearly dependent or linearly independent.

(b) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Again, let

$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$

consider $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 \quad + x_3 = 0$

$x_2 + x_3 = 0$

x_3 - free

There are nontrivial solutions

$$\vec{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The vectors are linearly dependent.

Example

Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

(c) $$\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}}, \underset{\vec{v}_4}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}} \right\}$$

Again, use a matrix A and consider $A\vec{x} = \vec{0}$

set

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad + \tfrac{1}{3}x_4 &= 0 \\ x_2 \quad + 2x_4 &= 0 \\ x_3 - \tfrac{2}{3}x_4 &= 0 \end{aligned}$$

x_4 is free

A solution to the homogeneous equation is

$$x_1 = \frac{-1}{3} x_4$$
$$x_2 = -2x_4$$
$$x_3 = \frac{2}{3} x_4$$
$$x_4 \text{ - free}$$

$$\vec{x} = x_4 \begin{bmatrix} -1/3 \\ -2 \\ 2/3 \\ 1 \end{bmatrix}$$

In terms of the vectors, $\vec{v}_4 = \frac{1}{3}\vec{v}_1 + 2\vec{v}_2 - \frac{2}{3}\vec{v}_3$

A linear dependence relation is

$$-\frac{1}{3}\vec{v}_1 - 2\vec{v}_2 + \frac{2}{3}\vec{v}_3 + \vec{v}_4 = \vec{0}$$

Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Since $\vec{w}$ is in Span$\{\vec{u}, \vec{v}\}$,

$\vec{w} = c_1\vec{u} + c_2\vec{v}$ for some scalars c_1 and c_2.

Rearranging this equation gives

$$-c_1\vec{u} - c_2\vec{v} + \vec{w} = \vec{0}$$

The coefficients in this equation are

$-c_1$, $-c_2$, and 1. Note, $1 \neq 0$

So we have a linear dependence relation

showing that $\{\vec{u}, \vec{v}, \vec{w}\}$ is

linearly dependent.

Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (You can easily verify this.)

However,

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \quad \text{i.e.} \quad \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},$$

so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

Two More Theorems

Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent.** That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

More vectors than entries in each vector.

Theorem: Any set of vectors that contains the zero vector is linearly **dependent.**

Determine if the set is linearly dependent or linearly independent

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

4 vectors in $\mathbb{R}^3$ $\quad 4 > 3$

they are <u>dependent</u>.

Determine if the set is linearly dependent or linearly independent

(b) $$\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}, \right\}$$

The set contains $\vec{0}$, it is dependent.