Math 5510 - Partial Differential Equations

Sturm-Liouville Problems

Part C

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Outline

Rayleigh Quotient

The **Sturm-Liouville Differential Equation** problem:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0.$$

Multiply by ϕ and integrate:

$$\int_a^b \left[\phi\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi^2\right] dx + \lambda \int_a^b \phi^2\sigma(x)dx = 0.$$

The **eigenvalue** satisfies:

$$\lambda = -\frac{\displaystyle\int_a^b \left[\phi\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi^2\right] dx}{\int_a^b \phi^2\sigma(x)dx}.$$

Rayleigh Quotient

Integrate the **eigenvalue** equation by parts:

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q(x)\phi^2\right]dx}{\int_a^b \phi^2\sigma(x)dx},$$

which is the **Rayleigh Quotient**.

The **eigenvalues** are nonnegative ($\lambda \geq 0$), if

1. $-p\phi\frac{d\phi}{dx}\Big|_a^b \geq 0$,
2. $q \leq 0$.

These conditions commonly hold for **Physical problems**, where $q \leq 0$ or ***energy-absorbing***.

Minimization Principle

The **eigenvalue** satisfies:

Theorem (Minimization Principle)

The minimum value of the ***Rayleigh quotient*** *for all continuous functions satisfying the* ***BCs*** *(not necessarily the differential equation) is the* ***lowest eigenvalue****:*

$$\lambda = \min_u \frac{-pu\frac{du}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{du}{dx}\right)^2 - q(x)u^2\right]dx}{\int_a^b u^2\sigma(x)dx},$$

This **minimum** occurs at $u = \phi_1$, the ***lowest eigenfunction***.

Trial functions

Trial functions: Cannot test all ***continuous functions*** satisfying the **BCs**, but select ***trial functions***, u_T,

$$\lambda_1 \le RQ[u_T] = \frac{-\,pu_T\frac{du_T}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{du_T}{dx}\right)^2 - q(x)u_T^2\right]dx}{\int_a^b u_T^2\sigma(x)dx},$$

This provides an ***upper bound*** for λ_1.

Example: Consider the **Sturm-Liouville** problem:

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi(1) = 0.$$

This example has an ***eigenvalue***, $\lambda_1 = \pi^2$, with an associated ***eigenfunction***, $\phi_1 = \sin(\pi x)$.

Trial functions

Example: We compute the **Rayleigh quotient** with **3** test functions, $u_1(x)$, $u_2(x)$, and $u_3(x)$:

Tent function:

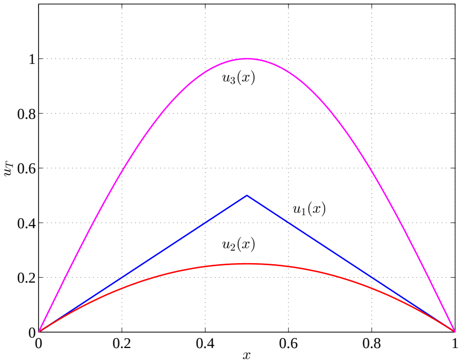
$$u_1(x) = \begin{cases} x, & x < \frac{1}{2}, \\ 1 - x, & x \geq \frac{1}{2}. \end{cases}$$

Quadratic function:

$$u_2(x) = x - x^2.$$

Eigenfunction:

$$u_3(x) = \sin(\pi x).$$



We insert each of these functions into the **Rayleigh quotient**.

Trial functions

Example: The **Rayleigh quotient** with

$$u_1(x) = \begin{cases} x, & x < \frac{1}{2}, \\ 1 - x, & x \geq \frac{1}{2}, \end{cases}$$

satisfies:

$$\begin{aligned} \lambda_1 \leq RQ[u_1] &= \frac{-u_1 \frac{du_1}{dx}\Big|_0^1 + \int_0^1 \left(\frac{du_1}{dx}\right)^2 dx}{\int_0^1 u_1^2 dx}, \\ &= \frac{\int_0^{1/2} dx + \int_{1/2}^1 dx}{\int_0^{1/2} x^2 dx + \int_{1/2}^1 (1-x)^2 dx}, \\ &= \frac{\frac{1}{2} + \frac{1}{2}}{\frac{1}{24} + \frac{1}{24}} = 12. \end{aligned}$$

Trial functions

Example: The **Rayleigh quotient** with $u_2(x) = x - x^2$ satisfies:

$$\lambda_1 \leq RQ[u_2] = \frac{-u_2 \frac{du_2}{dx}\Big|_0^1 + \int_0^1 \left(\frac{du_2}{dx}\right)^2 dx}{\int_0^1 u_2^2 dx},$$

$$= \frac{\int_0^1 (1-2x)^2 dx}{\int_0^1 (x-x^2)^2 dx} = \frac{1 - 2 + \frac{4}{3}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{5}} = 10.$$

The **Rayleigh quotient** with $u_3(x) = \sin(\pi x)$ satisfies:

$$\lambda_1 \leq RQ[u_3] = \frac{-u_3 \frac{du_3}{dx}\Big|_0^1 + \int_0^1 \left(\frac{du_3}{dx}\right)^2 dx}{\int_0^1 u_3^2 dx},$$

$$= \pi^2 \frac{\int_0^1 \cos^2(\pi x) dx}{\int_0^1 \sin^2(\pi x) dx}, = \frac{\pi^2 \frac{1}{2}}{\frac{1}{2}} = \pi^2 \approx 9.8696.$$

Rayleigh quotient

Proof: The proof of the **Rayleigh quotient** generally uses the **Calculus of Variations**, which cannot be developed here.

Our proof is based on ***eigenfunction expansion***.

We assume u is a ***continuous*** function satisfying ***homogeneous BCs***

Assuming ***homogeneous BCs*** gives the equivalent form for the **Rayleigh quotient**:

$$RQ[u] = \frac{-\int_a^b uL(u)dx}{\int_a^b u^2\sigma dx},$$

where L is the ***Sturm-Liouville operator***.

We take u expanded by the ***eigenfunctions***

$$u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x).$$

Rayleigh quotient

Proof (cont): Since L is a ***linear operator***, we expect

$$L(u) = \sum_{n=1}^{\infty} a_n L(\phi_n(x)) = -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \phi_n(x),$$

where later we show the interchange of the summation and operator when u is ***continuous*** and satisfies ***homogeneous BCs*** of the ***eigenfunctions***.

With different dummy summations, the **Rayleigh quotient** becomes

$$RQ[u] = \frac{\int_a^b \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m a_n \lambda_n \phi_m \phi_n \sigma\right) dx}{\int_a^b \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m a_n \phi_m \phi_n \sigma\right) dx}.$$

We interchange the summation and integration and use **orthogonality** to give

$$RQ[u] = \frac{\sum_{n=1}^{\infty} a_n^2 \lambda_n \int_a^b \phi_n^2 \sigma dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma dx}.$$

Rayleigh quotient

Proof: The previous equation gives the exact expression for the **Rayleigh quotient** in terms of the generalized **Fourier coefficients** a_n of u. If λ_1 is the lowest ***eigenvalue***, then we obtain:

$$RQ[u] \geq \frac{\lambda_1 \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma dx} = \lambda_1.$$

Note that equality holds only if $a_n = 0$ for $n > 1$, which gives the **minimization** result that $RQ[u] = \lambda_1$ for $u = a_1\phi_1$.

The proof is easily extended to show that if $a_1 = 0$ for the ***eigenfunction expansion*** of u, then $RQ[u] = \lambda_2$ when $a_n = 0$ for $n > 2$ and $u = a_2\phi_2$.

Thus, the **minimum** value for all continuous functions u that are ***orthogonal to the lowest eigenfunction*** and satisfy the ***homogeneous BCs*** is the next-to-lowest ***eigenvalue***.

Robin Boundary Conditions

Heat Equation with BC of Third Kind: Consider the **PDE**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with the **BCs**

$$u(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = -hu(L,t).$$

If $h > 0$, then this is a ***physical problem*** and the right endpoint represents **Newton's law of cooling** with an environmental temperature of $0°$.

Note: The problem solving below can be done equally well with the **String Equation**, $u_{tt} = c^2 u_{xx}$, where the right **BC** represents a restoring force for $h > 0$ and is called an ***elastic BC***.

If $h < 0$, either problem is not physical, as the **heat equation** would be having heat constantly pumped into the rod, and the **string equation** has a destabilizing force on the right end.

Robin Boundary Conditions

Separation of Variables: Let

$$u(x,t) = G(t)\phi(x),$$

then as before, the time dependent **ODEs** are

Heat Flow: $$\frac{dG}{dt} = -\lambda k G,$$

Vibrating String: $$\frac{d^2G}{dt^2} = -\lambda c^2 G.$$

The **Sturm-Liouville problem** becomes:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi'(L) + h\phi(L) = 0,$$

where $h \geq 0$ is ***physical*** and $h < 0$ is ***nonphysical***.

Robin Boundary Conditions

Positive eigenvalues: Let $\lambda = \alpha^2 > 0$, then

$$\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

The **BC**, $\phi(0) = 0$, implies $c_1 = 0$.

The other **BC**, $\phi'(L) + h\phi(L) = 0$, implies that $c_2\left(\alpha \cos(\alpha L) + h \sin(\alpha L)\right) = 0$ or

$$\tan(\alpha L) = -\frac{\alpha}{h} = -\frac{\alpha L}{hL}.$$

This is a ***transcendental equation*** in α, which cannot be solved exactly.

Robin Boundary Conditions

Eigenvalue equation is given by

$$\tan(\alpha L) = -\frac{\alpha L}{hL}, \qquad h > 0.$$

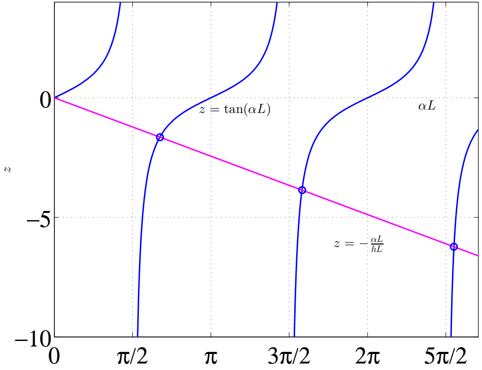
This equation can only be solved numerically, such as **Maple** or **MatLab**

This sketch is for the **physical** case, $h > 0$.

Visually, can see that asymptotically:

$$\alpha_n L \approx \left(n - \frac{1}{2}\right)\pi,$$

as $n \to \infty$



Robin Boundary Conditions

Again the **eigenvalue equation** is given by

$$\tan(\alpha L) = -\frac{\alpha L}{hL}, \qquad -1 < hL < 0.$$

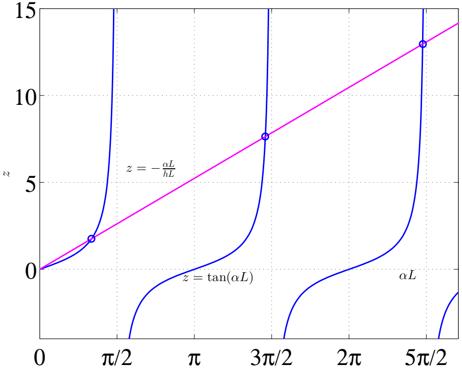
This sketch is for the **nonphysical** case, $-1 < hL < 0$, which is 1 of 3 cases.

There is a lowest ***eigenvalue***, $\lambda_1 < \frac{\pi}{2}$.

Asymptotically:

$$\alpha_n L \approx \left(n - \frac{1}{2}\right)\pi,$$

as $n \to \infty$

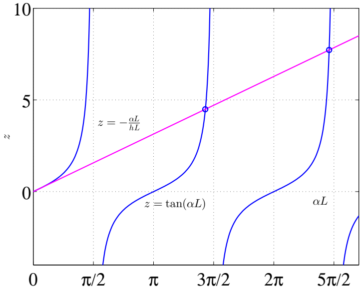


Robin Boundary Conditions

There are two additional cases for the **nonphysical problem**, where

$$\tan(\alpha L) = -\frac{\alpha L}{hL}, \qquad hL = -1 \quad \text{or} \quad hL < -1.$$

In both cases, the first ***positive eigenvalue*** satisfies $\pi < \lambda < \frac{3\pi}{2}$.



$h = -1$ $\qquad$ $h < -1$

Robin Boundary Conditions

The **nonphysical problem** with $hL = -1$ has its first ***positive eigenvalue***, $\alpha L \approx 4.49341$ ($\lambda = \alpha^2$).

Zero E.V.: Consider $\lambda = 0$, which gives the solution $\phi(x) = c_1 x + c_2$

The **BC** $\phi(0) = c_2 = 0$.

The other **BC**

$$\phi'(L) + h\phi(L) = c_1(1 + hL) = 0,$$

so if $hL = -1$, then $\lambda_0 = 0$ is an ***eigenvalue*** with associated ***eigenfunction***,

$$\phi_0(x) = x.$$

Robin Boundary Conditions

Negative E.V.: We don't expect negative ***eigenvalues*** for **physical problems**, as it produces an exponentially growing t-solution.

Suppose $\lambda = -\alpha^2 < 0$, so $\phi'' - \alpha^2 = 0$, which has the general solution:

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

The **BC** $\phi(0) = c_1 = 0$.

The remaining **BC** gives:

$$c_2 \left(\alpha \cosh(\alpha L) + h \sinh(\alpha L)\right) = 0,$$

which is nontrivial if

$$\tanh(\alpha L) = -\frac{\alpha}{h} = -\frac{\alpha L}{hL},$$

which is another ***transcendental equation***.

Robin Boundary Conditions

There are **4** cases to consider solving

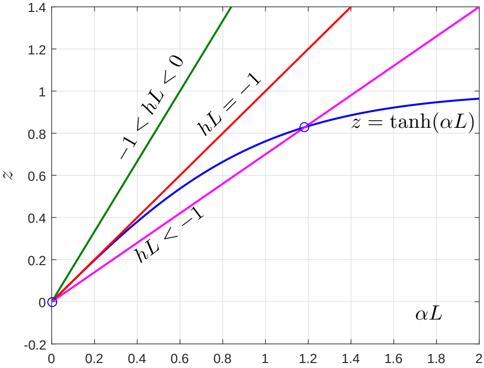
$$\tanh(\alpha L) = -\frac{\alpha L}{hL}.$$

Physical case $(hL > 0)$ has a negative slope, so only intersects origin.

When $-1 < hL < 0$, only intersects origin.

When $hL = -1$, line is tangent to origin.

When $hL < -1$, there is a ***unique positive eigenvalue***



Robin Boundary Conditions - Physical Problem

Heat Equation: Consider the **PDE**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with the **BCs**

$$u(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = -hu(L,t), \quad h > 0,$$

and **ICs**

$$u(x,0) = f(x).$$

The **Sturm-Liouville problem** had ***eigenvalues***, $\lambda_n = \alpha_n^2$, where $\alpha_n, \quad n = 1, 2, ...$ solves

$$\tan(\alpha_n L) = -\frac{\alpha_n L}{hL},$$

and corresponding ***eigenfunctions***

$$\phi_n = \sin(\alpha_n x).$$

Robin Boundary Conditions - Physical Problem

Heat Equation (cont): The time dependent solution is

$$G_n(t) = e^{-k\lambda_n t} = e^{-k\alpha_n^2 t}.$$

With the product solution, $u_n(x,t) = G_n(t)\phi_n(x)$, the **superposition principle** gives:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin(\alpha_n x),$$

where α_n satisfies $\tan(\alpha_n L) = -\frac{\alpha_n L}{hL}$.

The ***generalized Fourier coefficients*** satisfy:

$$A_n = \frac{\int_0^L f(x) \sin(\alpha_n x) dx}{\int_0^L \sin^2(\alpha_n x) dx}.$$

Robin Boundary Conditions - Physical Problem

Heat Equation (cont): However, with $\sin(\alpha_n L) = -\frac{\alpha_n}{h}\cos(\alpha_n L)$

$$\int_0^L \sin^2(\alpha_n x)dx = \frac{2\alpha_n L - \sin(2\alpha_n L)}{4\alpha_n} = \frac{Lh + \cos^2(\alpha_n L)}{2h}.$$

Thus, the ***generalized Fourier coefficients*** satisfy:

$$A_n = \frac{2h\displaystyle\int_0^L f(x)\sin(\alpha_n x)dx}{Lh + \cos^2(\alpha_n L)},$$

and the temperature in the rod is given by

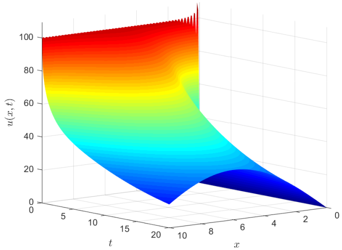
$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t}\sin(\alpha_n x).$$

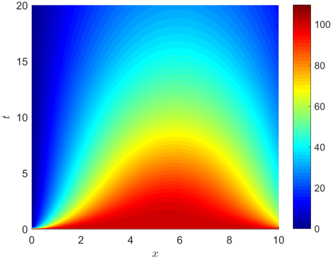
Robin Boundary Conditions - Physical Problem

Take $L = 10$, $k = 1$, and $h = 0.5$ and suppose $f(x) = 100$ for $0 \le x \le 10$. The Fourier coefficients are readily found:

$$A_n = \frac{200h\,(1-\cos(\alpha_n L))}{\alpha_n\,(Lh+\cos^2(\alpha_n L))}.$$

Solution with 100 terms.





Robin Boundary Conditions - Physical Problem

```
% Solutions to the heat flow equation
% on one-dimensional rod length L
% Right end with Robin Condition
format compact;
L = 10;                 % width of plate
Temp = 100;             % Constant temperature of ...
    rod, initially
tfin = 20;              % final time
k = 1;                  % heat coef of the medium
h = 0.5;                % Newton cooling constant
NptsX=151;              % number of x pts
NptsT=151;              % number of t pts
Nf=100;                  % number of Fourier terms
x=linspace(0,L,NptsX);
t=linspace(0,tfin,NptsT);
[X,T]=meshgrid(x,t);
```

Robin Boundary Conditions - Physical Problem

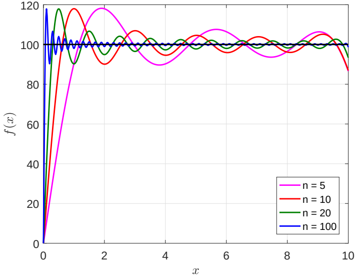
```
figure(1)
clf
a = zeros(1,Nf);
b = zeros(1,Nf);
U = zeros(NptsT,NptsX);
z0 = 2.7;
for n=1:Nf
    z0 = fsolve(@(x) h*L*sin(x)+x*cos(x),z0);
    a(n) = z0/L;
    b(n)=(2*Temp*h/(a(n)*(L*h+(cos(a(n)*L))^2)))...
        *(1-cos(a(n)*L)); % Fourier coefficients
    Un=b(n)*exp(-k*(a(n))^2*T).*sin(a(n)*X); % ...
        Temperature(n)
    U=U+Un;
    z0 = z0 + pi;
end
```

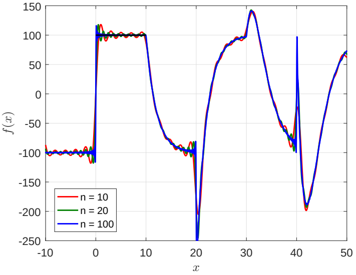
Robin Boundary Conditions - Physical Problem

```
set(gca,'FontSize',[12]);
surf(X,T,U);
shading interp
colormap(jet)
xlabel('$x$','Fontsize',12,'interpreter','latex');
ylabel('$t$','Fontsize',12,'interpreter','latex');
zlabel('$u(x,t)$','Fontsize',12,'interpreter','latex');
axis tight
view([141 10])
```

Fourier Series - BC 3^{rd} Kind

The solution of the **Heat Equation** with **Robin BCs** used the Fourier expansion of $f(x) = 100$ with the eigenfunctions, $\phi_n = \sin(\alpha_n x)$. Below are graphs showing the eigenfunction expansion.





Fourier Series - BC 3^{rd} Kind

```
% Fourier series
format compact;
L = 10;                 % width of plate
Temp = 100;             % Constant temperature of ...
    rod, initially
h = 0.5;                % Newton cooling constant
NptsX=500;              % number of x pts
Nf=100;                  % number of Fourier terms
X=linspace(0,L,NptsX);
a = zeros(1,Nf);
b = zeros(1,Nf);
U = zeros(1,NptsX);
U1 = zeros(1,NptsX);
U2 = zeros(1,NptsX);
U3 = zeros(1,NptsX);
z0 = 2.7;
```

Fourier Series - BC 3^{rd} Kind

```
for n=1:Nf
    z0 = fsolve(@(x) h*L*sin(x)+x*cos(x),z0);
    a(n) = z0/L;
    b(n)=(2*Temp*h/(a(n)*(L*h+(cos(a(n)*L))^2)))...
        *(1-cos(a(n)*L));  % Fourier coefficients
    Un = b(n)*sin(a(n)*X);  % Temperature(n)
    U = U+Un;
    if (n ≤ 5)
        U1 = U1+Un;
    end
    if (n ≤ 10)
        U2 = U2+Un;
    end
    if (n ≤ 20)
        U3 = U3+Un;
    end
    z0 = z0 + pi;
end
```

Fourier Series - BC 3^{rd} Kind

```
plot(X,U1,'m-','LineWidth',1.5);
hold on
plot(X,U2,'r-','LineWidth',1.5);
plot(X,U3,'-','Color',[0 0.5 0],'LineWidth',1.5);
plot(X,U,'b-','LineWidth',1.5);
plot([0 10],[100 100],'k-','LineWidth',1.5);
grid;
legend('n = 5','n = 10','n = 20','n = 100',...
    'location','southeast');
xlim([0 10]);
ylim([0 120]);
xlabel('$x$','Fontsize',12,'interpreter','latex');
ylabel('$f(x)$','Fontsize',12,'interpreter','latex');
set(gca,'FontSize',[12]);
```

Robin Boundary Conditions - Non-Physical Problem

Heat Equation with **Non-Physical BCs** satisfies:

PDE: $u_t = ku_{xx}$, **BC**: $u(0,t) = 0$,

IC: $u(x,0) = f(x)$, $u_x(L,t) = -hu(L,t)$ with $h < 0$.

For $-1 < h < 0$, the **Sturm-Liouville problem** is the same as the **physical problem** with ***eigenvalues***, $\lambda_n = \alpha_n^2$, where α_n, $n = 1, 2, ...$ solves $\tan(\alpha_n L) = -\frac{\alpha_n L}{hL}$, and corresponding ***eigenfunctions*** are

$$\phi_n = \sin(\alpha_n x).$$

The solution satisfies:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin(\alpha_n x),$$

with the same generalized Fourier coefficients as for the **physical problem**.

Robin Boundary Conditions - Non-Physical Problem

Heat Equation with **Non-Physical BCs** and $h = -1$ has $\lambda_0 = 0$ with the eigenfunction $\phi_0(x) = x$, so the solution becomes:

$$u(x,t) = A_0 x + \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin(\alpha_n x),$$

with A_n as before for $n = 1, 2, ...$ and

$$A_0 = \frac{3}{L^3} \int_0^L x f(x) dx.$$

If $h < -1$ and β_1 solves $\tanh(\beta_1 L) = -\frac{\beta_1}{h}$, then there is the additional eigenfunction $\phi_{-1}(x) = \sinh(\beta_1 x)$, so the solution becomes:

$$u(x,t) = A_{-1} e^{k\beta_1^2 t} \sinh(\beta_1 x) + \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin(\alpha_n x),$$

with A_n as before for $n = 1, 2, ...$ and

$$A_{-1} = \frac{2\beta_1 \int_0^L f(x) \sinh(\beta_1 x) dx}{\cosh(\beta_1 L) \sinh(\beta_1 L) - \beta_1 L}.$$

Robin Boundary Conditions - Physical Problem

Heat Equation with $h = 0$ (insulated right end) satisfies:

PDE: $u_t = ku_{xx}$, **BC**: $u(0,t) = 0$,

IC: $u(x,0) = f(x)$, $u_x(L,t) = 0$.

This problem is solved in the normal manner as before, and it is easy to see that the ***eigenvalues***, $\lambda_n = \frac{\left(n-\frac{1}{2}\right)^2\pi^2}{L^2}$, with corresponding ***eigenfunctions*** are

$$\phi_n = \sin\left(\frac{\left(n-\frac{1}{2}\right)\pi x}{L}\right).$$

The solution satisfies:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t} \sin\left(\frac{\left(n-\frac{1}{2}\right)\pi x}{L}\right),$$

with similar Fourier coefficients to our original **Heat problem**.

Eigenvalue Asymptotic Behavior

Examine the **Sturm-Liouville eigenvalue problem** in the form

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

The ***eigenvalues*** generally must be computed numerically.

There is a number of people working on details of these problems, so the scope of this problem is beyond this course. (See Mark Dunster)

Interpret this problem like a **spring-mass** problem for large λ, where x is time and ϕ is position.

- $p(x)$ acts like the mass.
- For λ large, $-\lambda\sigma(x)\phi$ acts like a restoring force
- This solution rapidly oscillates

Eigenvalue Asymptotic Behavior

With large λ, the solution oscillates rapidly over a few periods, so can approximate the coefficients as constants.

Thus, the DE is approximated near any point x_0 by

$$p(x_0)\frac{d^2\phi}{dx^2} + \lambda\sigma(x_0)\phi \approx 0,$$

which is like a standard **spring-mass** problem.

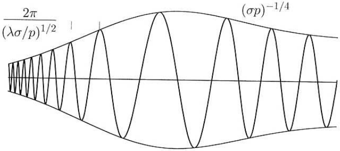
It follows that the frequency is approximated by

$$\omega = \sqrt{\frac{\lambda\sigma(x_0)}{p(x_0)}}$$

Eigenvalue Asymptotic Behavior

The ***amplitude*** and ***frequency*** are slow varying, so

$$\phi(x) = A(x)\cos(\psi(x)).$$



With Taylor series, we write

$$\phi(x) = A(x)\cos[\psi(x_0) + \psi'(x_0)(x - x_0) + ...],$$

so the ***local frequency*** is $\psi'(x_0)$, where

$$\psi\,'(x_0) = \lambda^{1/2}\left(\frac{\sigma(x_0)}{p(x_0)}\right)^{1/2}.$$

Eigenvalue Asymptotic Behavior

Integrating $\psi'(x_0)$ gives the correct phase

$$\psi(x) = \lambda^{1/2} \int^x \left(\frac{\sigma(x_0)}{p(x_0)} \right)^{1/2} dx_0.$$

It can be shown (beyond this class) that the independent solutions are approximated for large λ by

$$\phi(x) \approx (\sigma p)^{-1/4} exp \left[\pm i \lambda^{1/2} \int^x \left(\frac{\sigma}{p} \right)^{1/2} dx_0 \right].$$

If $\phi(0) = 0$, then the ***eigenfunction*** can be approximated by

$$\phi(x) = (\sigma p)^{-1/4} \sin \left(\lambda^{1/2} \int^x \left(\frac{\sigma}{p} \right)^{1/2} dx_0 \right) + \ldots$$

If the second BC is $\phi(L) = 0$, then

$$\lambda^{1/2} \int_0^L \left(\frac{\sigma}{p} \right)^{1/2} dx_0 \approx n\pi \qquad \text{or} \qquad \lambda \approx \left[\frac{n\pi}{\int_0^L \left(\frac{\sigma}{p} \right)^{1/2} dx_0} \right]^2.$$

Eigenvalue Asymptotic Behavior

Example: Consider the ***eigenvalue problem***

$$\frac{d^2\phi}{dx^2} + \lambda(1+x)\phi = 0,$$

with **BCs** $\phi(0) = 0$ and $\phi(1) = 0$.

Our approximation gives:

$$\lambda \approx \left[\frac{n\pi}{\int_0^1 (1+x_0)^{1/2} dx_0}\right]^2 = \frac{n^2\pi^2}{\left[\frac{2}{3}(1+x_0)^{3/2}\Big|_0^1\right]^2} = \frac{n^2\pi^2}{\frac{4}{9}(2^{3/2}-1)^2}.$$

n	Numerical	Formula
1	6.5484	6.6424
2	26.4649	26.5697
3	59.6742	59.7819
4	106.1700	106.2789
5	165.9513	165.0607
6	239.0177	239.1275
7	325.3691	325.4790

Approximation Properties

We claimed that any ***piecewise smooth function***, $f(x)$, can be represented by the ***generalized Fourier series*** of ***eigenfunctions***:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

By ***orthogonality with weight*** $\sigma(x)$ of the eigenfunctions

$$a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)dx}{\int_a^b \phi_n^2(x)\sigma(x)dx}.$$

Suppose we use a finite expansion,

$$f(x) \approx \sum_{n=1}^{M} \alpha_n \phi_n(x).$$

How do we choose α_n to obtain the best approximation?

Approximation Properties

How do we define the "best approximation?"

Definition (Mean-Square Deviation)

The standard measure of **Error** is the **mean-square deviation**, which is given by:

$$E = \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x) dx.$$

This deviation uses the weighting function, $\sigma(x)$.

It penalizes heavily for a large deviation on a small interval.

Approximation Properties

The best approximation solves the system:

$$\frac{\partial E}{\partial \alpha_i} = 0, \qquad i = 1, 2, ..., M.$$

or

$$0 = \frac{\partial E}{\partial \alpha_i} = -2 \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \phi_i(x) \sigma(x) dx, \qquad i = 1, 2, ..., M.$$

This would be complicated, except that we have mutual ***orthogonality*** of the $\phi_i(x)$'s, so

$$\int_a^b f(x) \phi_i(x) \sigma(x) dx = \alpha_i \int_a^b \phi_i^2(x) \sigma(x) dx.$$

Solving this system for α_i gives the α_i as the ***generalized Fourier coefficients***.

Approximation Properties

An alternate proof of this result shows that the ***minimum error*** is:

$$E = \int_a^b f^2 \sigma dx - \sum_{n=1}^{M} \alpha_n^2 \int_a^b \phi_n^2 \sigma dx.$$

This equation shows that as M increases, the **error** decreases.

Definition (Bessel's Inequality)

Since $E \geq 0$,

$$\int_a^b f^2 \sigma dx \geq \sum_{n=1}^{M} \alpha_n^2 \int_a^b \phi_n^2 \sigma dx.$$

More importantly, any ***Sturm-Liouville eigenvalue problem*** has an ***eigenfunction expansion*** of $f(x)$, which converges in the ***mean*** to $f(x)$.

Approximation Properties

The ***convergence in mean*** implies that

$$\lim_{M \to \infty} E = 0,$$

which gives the following:

Definition (Parseval's Equality)

Since $E \geq 0$,

$$\int_a^b f^2 \sigma dx = \sum_{n=1}^{\infty} \alpha_n^2 \int_a^b \phi_n^2 \sigma dx.$$

This inequality is a ***generalization of the Pythagorean theorem***, which important in showing **completeness**.