Section 1.8: Intro to Linear Transformations

Recall that the product $A\mathbf{x}$ is a linear combination of the columns of A—turns out to be a vector. If the columns of A are vectors in $\mathbb{R}^m$, and there are n of them, then

- A is an $m \times n$ matrix,
- the product $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbb{R}^n$, and
- the vector $\mathbf{b} = A\mathbf{x}$ is a vector in $\mathbb{R}^m$.

So we can think of A as an **object that acts** on vectors $\mathbf{x}$ in $\mathbb{R}^n$ (via the product $A\mathbf{x}$) to produce vectors $\mathbf{b}$ in $\mathbb{R}^m$.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition: A transformation T (a.k.a. **function** or **mapping**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

This function can be denoted using the symbols

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

which reads *"T maps $\mathbb{R}^n$ into $\mathbb{R}^m$."*

The notation

$$\mathbf{x} \mapsto T(\mathbf{x})$$

is read *"$\mathbf{x}$ maps to $T(\mathbf{x})$."*

" T of $\vec{x}$ "

Some relevant terms and notation include

- $\mathbb{R}^n$ is the **domain** and $\mathbb{R}^m$ is called the **codomain.**
- For $\mathbf{x}$ in the domain, $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under T.
- The collection of all images is called the **range.**
- The notation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ indicates that $\mathbb{R}^n$ is the domain and $\mathbb{R}^m$ is the codomain.
- If $T(\mathbf{x})$ is defined by multiplication by the $m \times n$ matrix A, we may denote this by $\mathbf{x} \mapsto A\mathbf{x}$.

Matrix Transformation Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$. Define the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by the mapping $T(\mathbf{x}) = A\mathbf{x}$.

(a) Find the image of the vector $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ under T.

$$T(\vec{u}) = A\vec{u}$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$$

(b) Determine a vector $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

Find $\vec{x}$ such that $T(\vec{x}) = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$,

i.e. solve

$$A\vec{x} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$$

We can use an augmented matrix

$$\begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 2 \\ x_2 = -2 \end{array}$$

So a solution $\vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$$

(c) Determine if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the range of T.

The question is, does there exist $\vec{x}$ such that

$T(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$?

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Using an augmented matrix

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This corresponds to the system

$$\begin{aligned} 1x_1 + 0x_2 &= 0 \\ 0x_1 + 1x_2 &= 0 \\ 0x_1 + 0x_2 &= 1 \end{aligned}$$

$\rightarrow$ $0=1$ is always false

The equation $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is inconsistent

Hence $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not in the range of T.

Linear Transformations

Definition: A transformation T is **linear** provided

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}$,$\mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

Every matrix transformation (e.g. $\mathbf{x} \mapsto A\mathbf{x}$) is a linear transformation. And it turns out that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be expressed in terms of matrix multiplication.

A Theorem About Linear Transformations:

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0},$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for scalars c, d and vectors **u**,**v**.

And in fact

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1 T(\mathbf{u}_1) + c_2 T(\mathbf{u}_2) + \cdots + c_k T(\mathbf{u}_k).$$

Example

Let r be a nonzero scalar. The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = r\mathbf{x}$$

is a linear transformation[1].
Show that T is a linear transformation.

We need to show that $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
and $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^2$ and
scalar c.

Let $\vec{u}$ and $\vec{v}$ be any vectors in $\mathbb{R}^2$.

$T(\vec{u}+\vec{v}) = r(\vec{u}+\vec{v})$

[1] It's called a **contraction** if $0 < r < 1$ and a **dilation** when $r > 1$.

$$= r\vec{u} + r\vec{v}$$

$$= T(\vec{u}) + T(\vec{v})$$

Let c be any scalar,

$$T(c\vec{u}) = rc\vec{u}$$

$$= cr\vec{u} = cT(\vec{u})$$

Both properties hold, hence T is a linear transformation.

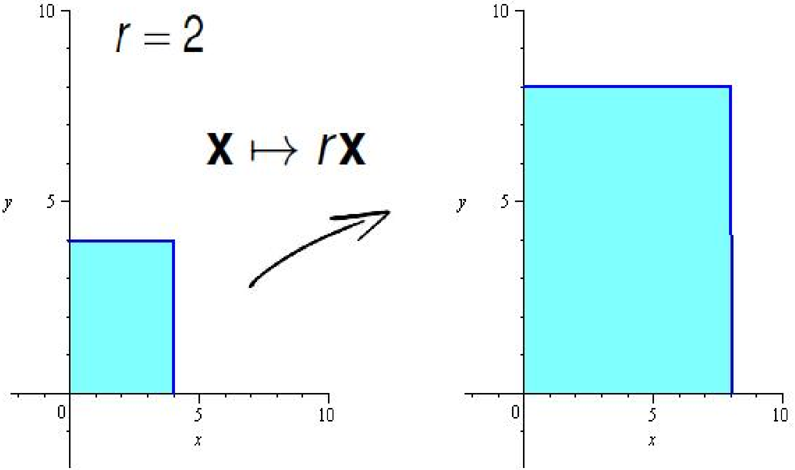


Figure: Geometry of dilation $\mathbf{x} \mapsto 2\mathbf{x}$. The 4 by 4 square maps to an 8 by 8 square.

Section 1.9: The Matrix for a Linear Transformation

Elementary Vectors: We'll use the notation $\mathbf{e}_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the i^{th} position and zero everywhere else.

e.g. in $\mathbb{R}^2$ the elementary vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

in $\mathbb{R}^3$ they would be

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and so forth.

Note that in $\mathbb{R}^n$, the elementary vectors are the columns of the identity I_n.

Matrix of Linear Transformation

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ be a linear transformation, and suppose

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}.$$

Use the fact that T is linear, and the fact that for each $\mathbf{x}$ in $\mathbb{R}^2$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

to find a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^2.$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2)$$

because T is Linear

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = \left[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n) \right].$$

The matrix A is called the **standard matrix** for the linear transformation T.

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the scaling trasformation (contraction or dilation for $r > 0$) defined by

$$T(\mathbf{x}) = r\mathbf{x}, \quad \text{for positive scalar } r.$$

Find the standard matrix for T. Calling it A, $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$.

$$T(\vec{e}_1) = r\vec{e}_1 = r\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}r\\0\end{bmatrix}$$

$$T(\vec{e}_2) = r\vec{e}_2 = r\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\r\end{bmatrix}$$

so $A = \begin{bmatrix}r & 0\\0 & r\end{bmatrix}$

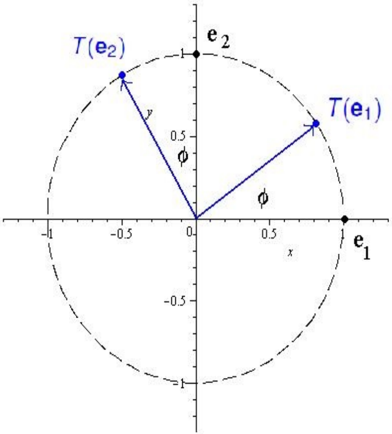
Note for $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ in $\mathbb{R}^2$

$$A\vec{x} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} r \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$= \begin{bmatrix} r x_1 \\ r x_2 \end{bmatrix} = r \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Another Graphics Example (Rotation)

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle ϕ.



Using some basic trigonometry, the points on the unit circle

$$T(\mathbf{e}_1) = (\cos\phi, \sin\phi)$$

$$T(\mathbf{e}_2) = (\cos(90^\circ + \phi), \sin(90^\circ + \phi))$$

$$= (-\sin\phi, \cos\phi)$$

So $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$.