Math 5510 - Partial Differential Equations

PDEs - Higher Dimensions

Part A

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Outline

Introduction

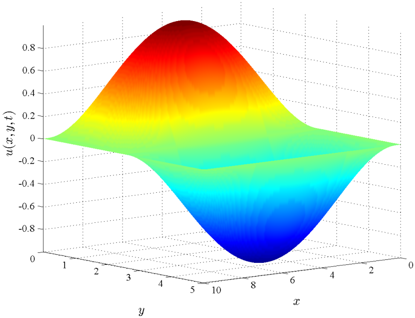
We want to consider **PDEs** in higher dimensions.

Vibrating Membrane:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Heat Conduction:

$$\frac{\partial u}{\partial t} = k \nabla^2 u$$



Rectangular Membrane

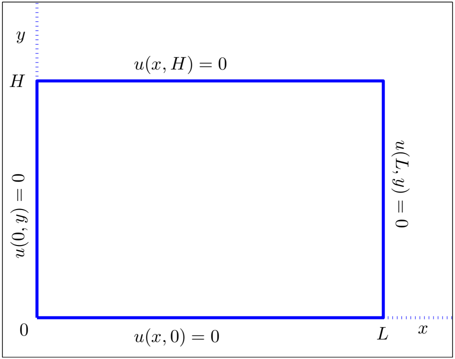
Vibrating Rectangular Membrane:

PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

BCs:

$$\begin{aligned} u(x,0,t) &= 0, \\ u(x,H,t) &= 0, \\ u(0,y,t) &= 0, \\ u(L,y,t) &= 0, \end{aligned}$$



ICs:

$$u(x,y,0) = \alpha(x,y) \quad \text{and} \quad u_t(x,y,0) = \beta(x,y).$$

Rectangular Membrane

Let $u(x, y, t) = h(t)\phi(x)\psi(y)$, then the **PDE** becomes

$$h''\phi\psi = c^2 \left(h\phi''\psi + h\phi\psi''\right).$$

This is rearranged to give

$$\frac{h''}{c^2 h} = \frac{\phi''}{\phi} + \frac{\psi''}{\psi} = -\lambda,$$

which gives the time dependent ODE:

$$h'' + \lambda c^2 h = 0.$$

The remaining ***spatial equation*** is rearranged to:

$$\phi'' + \psi'' = -\lambda\phi\psi \qquad \text{or} \qquad \frac{\phi''}{\phi} = -\frac{\psi''}{\psi} - \lambda = -\mu.$$

Rectangular Membrane

The ***spatial equations*** form two ***Sturm-Liouville problems***. With the **BCs** $u(0,y) = 0 = u(L,y)$, we obtain the 1^{st} ***Sturm-Liouville problem***:

$$\phi'' + \mu\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

From before, this gives the ***eigenvalues*** and ***eigenfunctions***:

$$\mu_m = \frac{m^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_m(x) = \sin\left(\frac{m\pi x}{L}\right).$$

If $\lambda - \mu_m = \nu$, then the 2^{nd} ***Sturm-Liouville problem*** is:

$$\psi'' + \nu\psi = 0, \qquad \psi(0) = 0 \quad \text{and} \quad \psi(H) = 0.$$

From before, this gives the ***eigenvalues*** and ***eigenfunctions***:

$$\nu_n = \frac{n^2\pi^2}{H^2} \qquad \text{and} \qquad \psi_n(y) = \sin\left(\frac{n\pi y}{H}\right).$$

Rectangular Membrane

From above we see $\lambda_{mn} = \mu_m + \nu_n = \frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{H^2} > 0$, so the time equation:

$$h'' + \lambda c^2 h = 0,$$

has the solution

$$h_{mn}(t) = a_n \cos(c\sqrt{\lambda_{mn}}t) + b_n \sin(c\sqrt{\lambda_{mn}}t).$$

The **Product solution** is

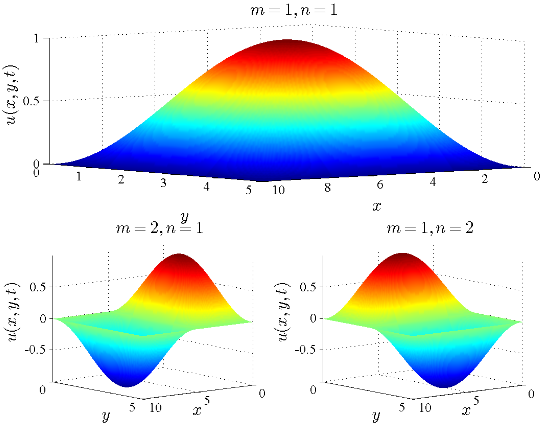
$$u_{mn}(t) = \left(a_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right) + b_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right).$$

The **Superposition Principle** gives

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(a_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right) + b_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right).$$

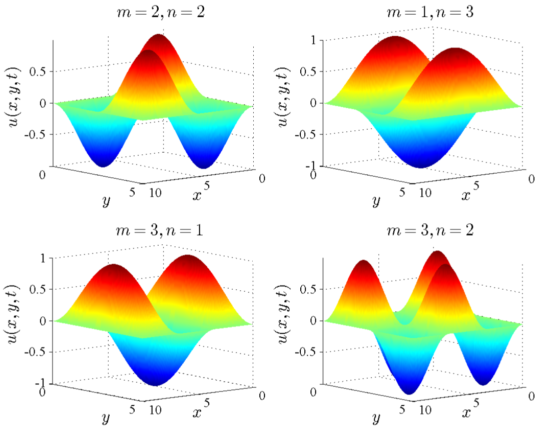
Nodal Curves

Nodal Curves



Nodal Curves

Nodal Curves



Rectangular Membrane

From the **ICs**, we have

$$u(x,y,0) = \alpha(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right).$$

Multiply by $\sin\left(\frac{j\pi x}{L}\right)$ and integrate $x \in [0, L]$ and $\sin\left(\frac{n\pi y}{H}\right)$ and integrate $y \in [0, H]$. **Orthogonality** gives:

$$a_{mn} = \frac{4}{LH}\int_0^H\int_0^L \alpha(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)dx\,dy.$$

Similarly,

$$u_t(x,y,0) = \beta(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} c\sqrt{\lambda_{mn}} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right),$$

and **orthogonality** gives:

$$b_{mn} = \frac{4}{LHc\sqrt{\lambda_{mn}}}\int_0^H\int_0^L \beta(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)dx\,dy.$$

Theorems for Eigenvalue Problems

Helmholtz Equation:

$$\nabla^2\phi + \lambda\phi = 0 \qquad \text{in } R,$$

with

$$\alpha\phi + \beta\nabla\phi\cdot\tilde{\mathbf{n}} = 0 \qquad \text{on } \partial R.$$

Generalizes to

$$\nabla\cdot(p\nabla\phi) + q\phi + \lambda\sigma\phi = 0.$$

Theorem

*1. All **eigenvalues** are real.*

*2. There exists infinitely many **eigenvalues** with a smallest, but no largest **eigenvalue**.*

*3. The may be many **eigenfunctions** corresponding to an **eigenvalue**.*

Theorems for Eigenvalue Problems

Theorem

4. The ***eigenfunctions*** *form a complete set, so if $f(x, y)$ is* ***piecewise smooth***

$$f(x, y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x, y).$$

5. ***Eigenfunctions*** *corresponding to different* ***eigenvalues*** *are orthogonal*

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \sigma dR = 0 \qquad \text{if} \quad \lambda_1 \neq \lambda_2.$$

Different ***eigenfunctions*** *belonging to the same* ***eigenvalue*** *can be made* ***orthogonal*** *by* ***Gram-Schmidt process****.*

Theorems for Eigenvalue Problems

Theorem

6. For $\sigma = 1$, an ***eigenvalue*** *λ can be related to the* ***eigenfunction*** *by the Rayleigh quotient:*

$$\lambda = \frac{-\oint_{\partial R} \phi \nabla \phi \cdot n \, ds + \iint_R |\nabla \phi|^2 dR}{\iint_R \phi^2 dR}.$$

The boundary conditions often simplify the boundary integral.

We use the **Example** for the ***vibrating rectangular membrane*** to illustrate a number of the Theorem results above.

Example

Example: The Sturm-Liouville problem for the ***vibrating rectangular membrane*** satisfies:

PDE: $\nabla^2\phi + \lambda\phi = 0,$

ICs: $\phi(0, y) = 0, \qquad \phi(L, y) = 0,$

$$\phi(x, 0) = 0, \qquad \phi(x, H) = 0.$$

We have already shown that this **Helmholtz equation** has ***eigenvalues***:

$$\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2, \qquad m = 1, 2, \ldots \quad n = 1, 2, \ldots$$

with corresponding ***eigenfunctions***:

$$\phi_{mn}(x, y) = \sin\left(\tfrac{m\pi x}{L}\right)\sin\left(\tfrac{n\pi y}{H}\right), \qquad m = 1, 2, \ldots \quad n = 1, 2, \ldots$$

Example

Example (cont): We already demonstrated that:

1. **Real eigenvalues**: The ***eigenvalues*** are clearly real.
2. **Ordering the eigenvalues**: It is easy to see that there is the lowest ***eigenvalue*** $\lambda_1 = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2$ and that there is no largest ***eigenvalue***, as m or $n \to \infty$.
3. **Multiple eigenvalues**: Suppose that $L = 2H$. It follows that

$$\lambda_{mn} = \frac{\pi^2}{4H^2}\left(m^2 + 4n^2\right).$$

It is easy to see for $m = 4, n = 1$ and $m = 2, n = 2$,

$$\lambda_{41} = \lambda_{22} = \frac{5\pi^2}{H^2}.$$

These solutions will oscillate with the same frequency.

Example

Example (cont): We have:

4. **Series of eigenfunctions**: If $f(x,y)$ is ***piecewise smooth***, then

$$f(x,y) \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

5. **Convergence**: As before, write the **Error** using a finite series

$$E = \iint_R \left(f - \sum_{\lambda} a_\lambda \phi_\lambda\right).$$

The approximation improves with increasing λ, and we found that the series $\sum_\lambda a_\lambda \phi_\lambda$ ***converges in the mean*** to f.

Orthogonality

Orthogonality: Assume $\lambda_1 \neq \lambda_2$ with ***eigenfunctions*** ϕ_{λ_1} and ϕ_{λ_2} and insert these into the equation:

$$\nabla \cdot (p \nabla \phi) + q\phi + \lambda \sigma \phi = 0.$$

Multiplying by the other eigenfunction and subtracting, we can write

$$\phi_{\lambda_1} \left(\nabla \cdot (p \nabla \phi_{\lambda_2}) \right) - \phi_{\lambda_2} \left(\nabla \cdot (p \nabla \phi_{\lambda_1}) \right) = (\lambda_2 - \lambda_1) \sigma \phi_{\lambda_1} \phi_{\lambda_2}.$$

Use integration by parts over the entire region R and the homogeneous boundary conditions to give (more details next section):

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \sigma dR = 0, \qquad \text{if} \quad \lambda_1 \neq \lambda_2.$$

Fourier Coefficients

Fourier Coefficients: Assume that f is ***piecewise smooth***, so

$$f(x,y) \sim \sum_{\lambda} a_\lambda \phi_\lambda.$$

Use the ***orthogonality relationship*** with respect to the weighting function σ:

$$\iint_R \phi_{\lambda_1}\phi_{\lambda_2}\sigma dR = 0, \qquad \text{if} \quad \lambda_1 \neq \lambda_2,$$

then the **Fourier coefficients** satisfy

$$a_{\lambda_i} = \frac{\iint_R f\phi_{\lambda_i}\sigma dR}{\iint_R \phi_{\lambda_i}^2 \sigma dR}.$$

Note: If there is more than one ***eigenfunction*** associated with an ***eigenvalue***, then assume the ***eigenfunctions*** have been made ***orthogonal*** by ***Gram-Schmidt***.

Green's Formula

Consider the **PDE**:

$$\nabla^2\phi + \lambda\phi = 0, \qquad \text{in} \quad R,$$

with **BCs**:

$$\beta_1\phi + \beta_2\nabla\phi \cdot \tilde{\mathbf{n}} = 0, \qquad \text{on} \quad \partial R,$$

where β_1 and β_2 are real functions in R.

Basic product rule gives:

$$\begin{aligned} \nabla \cdot (u\nabla v) &= u\nabla^2 v + \nabla u \cdot \nabla v, \\ \nabla \cdot (v\nabla u) &= v\nabla^2 u + \nabla v \cdot \nabla u. \end{aligned}$$

Subtracting gives:

$$u\nabla^2 v - v\nabla^2 u = \nabla \cdot (u\nabla v - v\nabla u).$$

Green's Formula

The previous result is integrated to give:

$$\iint_R \left(u\nabla^2 v - v\nabla^2 u\right) dR = \iint_R \nabla \cdot (u\nabla v - v\nabla u) dR.$$

Apply the **Divergence Theorem** and obtain:
Green's Formula: Also, **Green's second identity**:

$$\iint_R \left(u\nabla^2 v - v\nabla^2 u\right) dR = \oint_{\partial R} \left(u\nabla v - v\nabla u\right) \cdot \tilde{\mathbf{n}}\, dS.$$

This identity is important in showing an operator is **self-adjoint** if there are ***homogeneous BCs***.

Self-Adjoint Operator

Let $L = \nabla^2$ be a linear operator:

Theorem (Self-Adjoint)

If u and v are two functions such that

$$\oint_{\partial R} (u\nabla v - v\nabla u) \cdot \tilde{\mathbf{n}} \, dS = 0,$$

then

$$\iint_R \left(u\nabla^2 v - v\nabla^2 u\right) dR = \iint_R (uL[v] - vL[u]) \, dR = 0.$$

Note: The above theorem is stated in 2D, but it equally applies to 3D by substituting double integrals with triple integrals and line integrals with surface integrals.

Orthogonality

Orthogonality of Eigenfunctions: We use **Green's formula** to show ***orthogonality*** of ***eigenfunctions***, ϕ_1 and ϕ_2, corresponding to different ***eigenvalues***, λ_1 and λ_2.

Suppose with $L = \nabla^2$

$$L[\phi_1] + \lambda_1 \phi_1 = 0 \qquad \text{and} \qquad L[\phi_2] + \lambda_2 \phi_2 = 0.$$

If ϕ_1 and ϕ_2 satisfy the same ***homogeneous BCs***,

$$\oint_{\partial R} (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \tilde{\mathbf{n}}\, dS = 0,$$

then by **Green's formula**:

$$\iint_R (\phi_1 L[\phi_2] - \phi_2 L[\phi_1])\, dR = 0.$$

Orthogonality

However,

$$\iint_R (\phi_1 L[\phi_2] - \phi_2 L[\phi_1])\, dR = \iint_R (\lambda_2 \phi_1 \phi_2 - \lambda_1 \phi_1 \phi_2)\, dR$$

$$= (\lambda_2 - \lambda_1) \iint_R \phi_1 \phi_2 dR = 0.$$

So for $\lambda_2 \neq \lambda_1$, the ***eigenfunctions*** are **orthogonal**:

$$\iint_R \phi_1 \phi_2 dR = 0.$$

Gram-Schmidt Process

Gram-Schmidt Process: Suppose that ϕ_1, ϕ_2, ..., ϕ_m, are independent ***eigenfunctions*** all corresponding to the ***eigenvalue***, λ (a **single e.v.**).

Let $\psi_1 = \phi_1$ be an ***eigenfunction***.

Any linear combination of ***eigenfunctions*** is also an ***eigenfunction***, so take

$$\psi_2 = \phi_2 + c\psi_1.$$

We want

$$\iint_R \psi_1 \psi_2 dR = 0 = \iint_R \psi_1(\phi_2 + c\psi_1) dR,$$

so choose

$$c = -\frac{\iint_R \phi_2 \psi_1 dR}{\iint_R \psi_1^2 dR}.$$

Gram-Schmidt Process

Gram-Schmidt Process: Continuing take

$$\psi_3 = \phi_3 + c_1\psi_1 + c_2\psi_2.$$

We want

$$\begin{aligned} \iint_R \psi_3 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dR &= 0, \\ \iint_R (\phi_3 + c_1\psi_1 + c_2\psi_2) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dR &= 0. \end{aligned}$$

It follows that

$$c_1 = -\frac{\iint_R \phi_3\psi_1 dR}{\iint_R \psi_1^2 dR} \quad \text{and} \quad c_2 = -\frac{\iint_R \phi_3\psi_2 dR}{\iint_R \psi_2^2 dR}.$$

Gram-Schmidt Process

Gram-Schmidt Process: In general,

$$\psi_j = \phi_j - \sum_{i=1}^{j-1} \frac{\iint_R \phi_j \psi_i dR}{\iint_R \psi_i^2 dR} \psi_i.$$

Thus, we can always obtain an ***orthogonal set of eigenfunctions***.