Math 3100

Section 1.9: The Matrix for a Linear Transformation

Recall that **Definition:** A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$is **linear** provided

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, this matrix, called the **standard matrix** for the linear transformation T is

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)] .$$

Example[1]

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection transformation that projects each point onto the x_1 axis

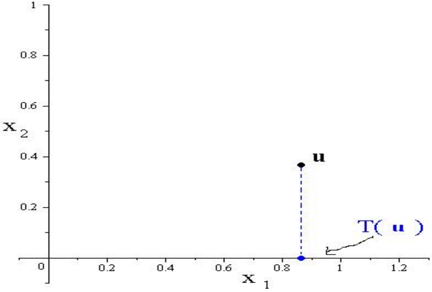
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

Find the standard matrix for T.

we need $T(\vec{e}_1)$

and $T(\vec{e}_2)$.

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



[1] See pages 73–75 in Lay for matrices associated with other geometric tranformation on $\mathbb{R}^2$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \left[T(\vec{e}_1) \;\; T(\vec{e}_2) \right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Note $\quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

The Property **Onto**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an **onto** transformation, then the equation

$$T(\mathbf{x}) = \mathbf{b}$$

is always solvable. If T is a linear transformation with standard matrix A, then this is equivalent to saying $A\mathbf{x} = \mathbf{b}$ is always consistent.

Determine if the transformation is onto.

Note $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

Let $\vec{b}$ in $\mathbb{R}^m$ be arbitrary. Here $m = 2$.

Consider $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and the matrix equation

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$ which is in rref.

The fourth column can not be a pivot column. Hence the equation $A\vec{x} = \vec{b}$ is always consistent.

Hence T is onto.

The Property **One to One**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **one to one** transformation, then the equation

$$T(\mathbf{x}) = T(\mathbf{y}) \quad \text{is only true when} \quad \mathbf{x} = \mathbf{y}.$$

Determine if the transformation is one to one.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

Letting $\vec{b}$ be in $\mathbb{R}^2$, we had the rref for $T(\vec{x}) = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}.$$

we can read the solution set

$$\begin{aligned} x_1 \qquad + 2x_3 &= b_1 \\ x_2 + 3x_3 &= b_2 \end{aligned} \implies \begin{aligned} x_1 &= b_1 - 2x_3 \\ x_2 &= b_2 - 3x_3 \\ x_3 &- \text{free} \end{aligned}$$

Any vector $\vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

satisfies $T(\vec{x}) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. The system has infinitely many solutions, hence

T is <u>not</u> one to one.

Some Theorems on Onto and One to One

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

Note $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_2 \end{bmatrix}$$

Find the matrix A:

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$

Let's verify that $T(\vec{x}) = A\vec{x}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_2 \end{bmatrix}$$

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ 2x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_2 \end{bmatrix} \checkmark$$

To show that T is one to one, we can show that $A\vec{x} = \vec{0}$ has only the trivial solution.

$$A\vec{x} = \vec{0} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0$$
$$x_2 = 0$$

$A\vec{x} = \vec{0}$ has only the trivial solution. T is one to one.

As for whether T is onto, let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

and consider $A\vec{x} = \vec{b}$. The augmented matrix is

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & b_3 + 3b_2 - 6b_1 \end{bmatrix}$$

The system is inconsistent if

$$b_3 + 3b_2 - 6b_1 \neq 0.$$

T is not onto.