Math 5510 - Partial Differential Equations

PDEs - Higher Dimensions
Cylinder

Ahmed Kaffel,
⟨ahmed.kaffel@marquette.edu⟩

Marquette University

https://www.mscsnet.mu.edu/~ahmed/teaching.html

Spring 2021

Outline

More on Bessel Functions

Bessel's Equation can be written:

$$\frac{d^2\phi}{dz^2} = -\left(1 - \frac{m^2}{z^2}\right)\phi - \frac{1}{z}\frac{d\phi}{dz},$$

which can be compared to the **damped-spring-mass** system:

$$\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}.$$

1. Bessel's equation behaves like a time-varying frictional force ($c \sim 1/t$) that gets weaker with time (less than exponential decay).
2. Bessel's equation behaves like a restoring force ($k \sim (1 - m^2/z^2)$) approaches constant oscillation.

More on Bessel Functions

Asymptotic Behavior of Bessel's Equation

Small z

$$J_0(z) \approx 1 \qquad Y_0(z) \approx \frac{2}{\pi}\ln(z)$$

$$J_1(z) \approx \frac{1}{2}z \qquad Y_1(z) \approx -\frac{2}{\pi}z^{-1}$$

$$J_2(z) \approx \frac{1}{8}z^2 \qquad Y_2(z) \approx -\frac{4}{\pi}z^{-2}$$

Large z, as $z \to \infty$

$$J_m(z) \approx \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right)$$

$$Y_m(z) \approx \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right)$$

The ***zeroes*** are asymptotically separated by π.

Vibrating Circular Membrane

Laplace's Equation - Cylinder: The PDE satisfies:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

BC: Bottom

$$u(r,\theta,0) = \alpha(r,\theta),$$

BC: Top

$$u(r,\theta,H) = \beta(r,\theta),$$

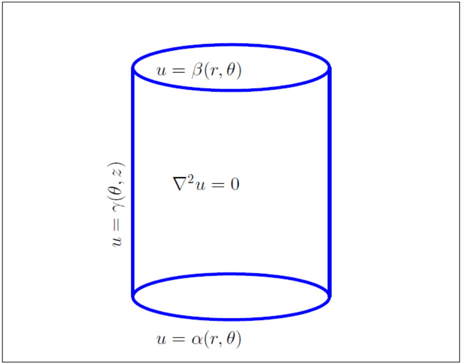
BC: Side

$$u(a,\theta,z) = \gamma(\theta,z).$$

BC: Implicit (Homogeneous)

Periodic in θ and
Bounded $r \to 0$.

Break the problem into
3 problems each with
2 homogeneous conditions.



Laplace's Equation - Cylinder

Problem 1: Let the **Top** and **Side** be **homogeneous** with only the **nonhomogeneous** condition:

$$u_1(r,\theta,0) = \alpha(r,\theta).$$

The boundedness as $r \to 0$ and periodicity in the θ direction provides the other homogeneous conditions.

Use **Separation of Variables** in **Laplace's Equation** with:

$$u_1(r,\theta,z) = \phi(r)g(\theta)h(z),$$

so

$$\frac{gh}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{\phi h}{r^2}\frac{d^2 g}{d\theta^2} + \phi g\frac{d^2 h}{dz^2} = 0.$$

Laplace's Equation - Cylinder

Separation of Variables gives

$$\frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)+\frac{1}{r^2 g}\frac{d^2g}{d\theta^2}=-\frac{h''}{h}=-\lambda,$$

which gives the z-equation:

$$h''-\lambda h=0.$$

Multiply by r^2 and rearrange to obtain:

$$\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)+\lambda r^2=-\frac{g''}{g}=\mu, \qquad \text{or} \qquad g''+\mu g=0.$$

Laplace's Equation - Cylinder

1^{st} **Sturm-Liouville Problem** is:

$$g'' + \mu g = 0, \qquad \text{with} \quad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi).$$

As seen before, this problem has ***eigenvalues***, $\mu_m = m^2$, $m = 0, 1, 2, ...$ and corresponding ***eigenfunctions***:

$$g_0(\theta) = a_0 \quad \text{and} \quad g_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta).$$

2^{nd} **Sturm-Liouville Problem** is:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0, \quad \text{with} \quad \phi(a) = 0 \quad \text{and} \quad |\phi(0)| < \infty,$$

which is **Bessel's equation of order m**.

Laplace's Equation - Cylinder

The 2^{nd} **Sturm-Liouville Problem** in r has the general solution:

$$\phi(r) = c_1 J_m\left(\sqrt{\lambda}r\right) + c_2 Y_m\left(\sqrt{\lambda}r\right).$$

Since $|\phi(0)| < \infty$, we have $c_2 = 0$. The other ***homogeneous BC*** gives:

$$\phi(a) = c_1 J_m\left(\sqrt{\lambda_{mn}}a\right) = 0.$$

As seen before, this has ***eigenvalues*** and ***eigenfunctions***;

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2, \qquad \phi_{mn}(r) = J_m(z_{mn}r/a), \qquad m = 0, 1, 2, \dots \quad n = 1, 2, \dots,$$

where z_{mn} is the n^{th} zero satisfying $J_m(z_{mn}) = 0$.

Laplace's Equation - Cylinder

With $\lambda_{mn} > 0$, we solve

$$h'' - \lambda h = 0,$$

to obtain

$$h(z) = d_1 \cosh\left(\sqrt{\lambda_{mn}}(H - z)\right) + d_2 \sinh\left(\sqrt{\lambda_{mn}}(H - z)\right).$$

However, $h(H) = 0$, so $d_1 = 0$ or $h(z) = \sinh\left(\sqrt{\lambda_{mn}}(H - z)\right)$.

We apply the **superposition principle** to obtain u_1:

$$\begin{aligned} u_1(r,\theta,z) \quad = \quad & \sum_{n=1}^{\infty} A_{0n} J_0\left(\sqrt{\lambda_{0n}}r\right) \sinh\left(\sqrt{\lambda_{0n}}(H - z)\right) + \\ & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\Big) \\ & \cdot J_m\left(\sqrt{\lambda_{mn}}r\right) \sinh\left(\sqrt{\lambda_{mn}}(H - z)\right). \end{aligned}$$

Laplace's Equation - Cylinder

Fourier coefficients are found with the ***nonhomogeneous BC***:

$$\begin{aligned} u_1(r,\theta,0) = \alpha(r,\theta) &= \sum_{n=1}^{\infty} A_{0n} J_0\left(\sqrt{\lambda_{0n}}r\right)\sinh\left(\sqrt{\lambda_{0n}}H\right) + \\ &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\Big) \\ &\cdot J_m\left(\sqrt{\lambda_{mn}}r\right)\sinh\left(\sqrt{\lambda_{mn}}H\right). \end{aligned}$$

With *orthogonality*, we find

$$A_{0n} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) J_0\left(\sqrt{\lambda_{0n}}r\right) r\,dr\,d\theta}{2\pi\sinh\left(\sqrt{\lambda_{0n}}H\right)\int_0^a J_0^2\left(\sqrt{\lambda_{0n}}r\right) r\,dr},$$

and

$$A_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\cos(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r\,dr\,d\theta}{\pi\sinh\left(\sqrt{\lambda_{mn}}H\right)\int_0^a J_m^2\left(\sqrt{\lambda_{mn}}r\right) r\,dr},$$

Laplace's Equation - Cylinder

and

$$B_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\sin(m\theta)J_m\left(\sqrt{\lambda_{mn}}r\right)r\,dr\,d\theta}{\pi\sinh\left(\sqrt{\lambda_{mn}}H\right)\int_0^a J_m^2\left(\sqrt{\lambda_{mn}}r\right)r\,dr}.$$

It is easy to see that almost identical computations hold for u_2 where the ***nonhomogeneous BC*** is the top, $u_2(r,\theta,H) = \beta(r,\theta)$.

The **2 Sturm-Liouville problems** are identical to the ones for u_1, so the only difference is solving the z-dependent equation:

$$h'' - \lambda_{mn}h = 0, \qquad \text{with} \quad h(0) = 0.$$

This has the solution:

$$h(z) = c_1\sinh\left(\sqrt{\lambda_{mn}}z\right).$$

Laplace's Equation - Cylinder

It follows that

$$\begin{aligned} u_2(r,\theta,z) &= \sum_{n=1}^{\infty} C_{0n} J_0\left(\sqrt{\lambda_{0n}}r\right)\sinh\left(\sqrt{\lambda_{0n}}z\right) + \\ & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)\Big) J_m\left(\sqrt{\lambda_{mn}}r\right)\sinh\left(\sqrt{\lambda_{mn}}z\right). \end{aligned}$$

The ***Fourier coefficients*** from the condition $\beta(r,\theta)$ are:

$$C_{0n} = \frac{\int_{-\pi}^{\pi}\int_0^a \beta(r,\theta) J_0\left(\sqrt{\lambda_{0n}}r\right) r\,dr\,d\theta}{2\pi\sinh\left(\sqrt{\lambda_{0n}}H\right)\int_0^a J_0^2\left(\sqrt{\lambda_{0n}}r\right) r\,dr},$$

and

$$C_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \beta(r,\theta)\cos(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r\,dr\,d\theta}{\pi\sinh\left(\sqrt{\lambda_{mn}}H\right)\int_0^a J_m^2\left(\sqrt{\lambda_{mn}}r\right) r\,dr},$$

and

$$D_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \beta(r,\theta)\sin(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r\,dr\,d\theta}{\pi\sinh\left(\sqrt{\lambda_{mn}}H\right)\int_0^a J_m^2\left(\sqrt{\lambda_{mn}}r\right) r\,dr}.$$

Laplace's Equation - Cylinder

The **cylinder problem** for u_3, where the ***nonhomogeneous BC*** is the side, $u_3(a, \theta, z) = \gamma(\theta, z)$, must be handled differently.

With the side nonhomogeneous, the r-dependent equation can no longer be one of the **2 Sturm-Liouville problems**.

The ***separation of variables*** for $u_3(r, \theta, z) = \phi(r)g(\theta)h(z)$ gives:

$$\frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{1}{r^2 g}\frac{d^2 g}{d\theta^2} = -\frac{h''}{h} = \lambda.$$

Now the 1^{st} **Sturm-Liouville problem** is:

$$h'' + \lambda h = 0, \qquad \text{with} \quad h(0) = 0 \quad \text{and} \quad h(H) = 0.$$

From before, this has the ***eigenvalues*** and ***eigenfunctions***:

$$\lambda_n = \frac{n^2\pi^2}{H^2} \qquad \text{with} \qquad h_n(z) = \sin\left(\frac{n\pi z}{H}\right).$$

Laplace's Equation - Cylinder

Multiplying by r^2 and rearranging the **separation equation** gives:

$$\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \lambda_n r^2 = -\frac{g''}{g} = \mu, \qquad \text{or} \qquad g'' + \mu g = 0.$$

The 2^{nd} **Sturm-Liouville Problem** is now:

$$g'' + \mu g = 0, \qquad \text{with} \quad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi),$$

which as before has ***eigenvalues***, $\mu_m = m^2$, $m = 0, 1, 2, \ldots$ and corresponding ***eigenfunctions***:

$$g_0(\theta) = a_0 \quad \text{and} \quad g_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta).$$

Laplace's Equation - Cylinder

Returning to the ***separation equation***, we obtain the 3^{rd} **ODE**, which is given by:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \left(\frac{n^2\pi^2}{H^2}r + \frac{m^2}{r}\right)\phi = 0, \quad \text{with} \quad |\phi(0)| < \infty,$$

which because of the sign is **not** **Bessel's equation**.

Let $z = \frac{n\pi}{H}r$, then the 3^{rd} **ODE** can be written:

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} - (z^2 + m^2)\phi = 0,$$

which is known as **modified Bessel's equation**.

This has the solution:

$$\phi(r) = c_1 K_m\left(\tfrac{n\pi}{H}r\right) + c_2 I_m\left(\tfrac{n\pi}{H}r\right).$$

The condition that $|\phi(0)| < \infty$ implies that $c_1 = 0$, as $K_m(z) \to \infty$ as $z \to 0$. ($I_m(z)$ behaves as z^m as $z \to 0$.)

Laplace's Equation - Cylinder

The **superposition principle** gives

$$\begin{aligned} u_3(r,\theta,z) &= \sum_{n=1}^{\infty} E_{0n} I_0\left(\tfrac{n\pi}{H}r\right)\sin\left(\tfrac{n\pi}{H}z\right) + \\ &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(E_{mn}\cos(m\theta) + F_{mn}\sin(m\theta)\Big) I_m\left(\tfrac{n\pi}{H}r\right)\sin\left(\tfrac{n\pi}{H}z\right). \end{aligned}$$

The ***Fourier coefficients*** from the condition $\gamma(\theta,z)$ are:

$$E_{0n} = \frac{\int_{-\pi}^{\pi}\int_0^H \gamma(\theta,z)\sin\left(\frac{n\pi}{H}z\right)dz\,d\theta}{\pi H I_0\left(\frac{n\pi}{H}a\right)},$$

and

$$E_{mn} = \frac{2\int_{-\pi}^{\pi}\int_0^H \gamma(\theta,z)\cos(m\theta)\sin\left(\frac{n\pi}{H}z\right)dz\,d\theta}{\pi H I_m\left(\frac{n\pi}{H}a\right)},$$

and

$$F_{mn} = \frac{2\int_{-\pi}^{\pi}\int_0^H \gamma(\theta,z)\sin(m\theta)\sin\left(\frac{n\pi}{H}z\right)dz\,d\theta}{\pi H I_m\left(\frac{n\pi}{H}a\right)}.$$

Modified Bessel Functions

Modified Bessel's functions satisfy:

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} - (z^2+m^2)\phi = 0,$$

We could write this equation:

$$\frac{d^2\phi}{dz^2} = -\frac{1}{z}\frac{d\phi}{dz} + \left(1+\frac{m^2}{z^2}\right)\phi,$$

which for large z gives:

$$\frac{d^2\phi}{dz^2} \approx \phi.$$

This ***differential equation*** has solutions, like e^x and e^{-x}.

In fact, it can be shown that only one ***linearly independent solution*** decays as $z \to \infty$, and we define this solution:

$$K_m(z) \sim \sqrt{\frac{\pi}{2}}\frac{e^{-z}}{z^{1/2}}.$$

Modified Bessel Functions

However, $K_m(z)$ is ***singular*** as $z \to 0$, and it can be shown that

$$K_m(z) \sim \begin{cases} \ln(z), & m = 0, \\ \frac{1}{2}(m-1)!\left(\frac{1}{2}z\right)^{-m}, & m \neq 0. \end{cases}$$

So significantly, $K_m(z)$ ***decays exponentially*** as $z \to \infty$, but is ***singular*** as $z \to 0$.

The ***Modified Bessel Function*** is uniquely defined such that

$$I_m(z) \sim \frac{1}{m!}\left(\frac{1}{2}z\right)^m,$$

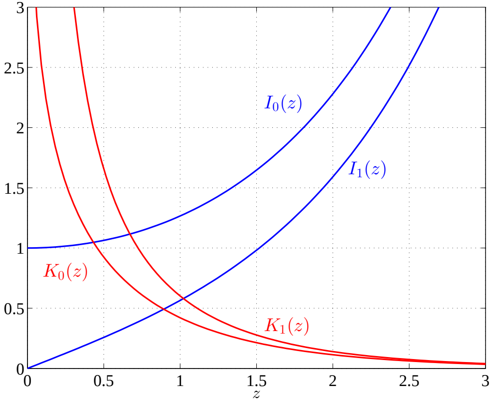
as $z \to 0$.

However, as $z \to \infty$, it is a linear combination of the independent solutions, which behave like

$$I_m(z) \sim \sqrt{\frac{1}{2\pi z}}e^z.$$

Modified Bessel Functions

So significantly, $I_m(z)$ ***grows exponentially*** as $z \to \infty$, but is well-behaved at $z = 0$. Below is the graph of some of the ***modified Bessel functions***.



Spherical Problems

The **Heat** or **Wave** equations:

$$\frac{\partial u}{\partial t} = k\nabla^2 u \qquad \text{or} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u,$$

can use the ***separation of variables*** $u(\rho, \theta, \phi, t) = w(\rho, \theta, \phi)h(t)$ to obtain either

$$\frac{h'}{kh} = \frac{\nabla^2 w}{w} = -\lambda \qquad \text{or} \qquad \frac{h''}{c^2 h} = \frac{\nabla^2 w}{w} = -\lambda.$$

Thus, we have the ***time-equation***:

$$h' + \lambda k h = 0 \qquad \text{or} \qquad h'' + \lambda c^2 h = 0.$$

The ***space-equation*** is:

$$\nabla^2 w + \lambda w = 0.$$

Spherical Problems

In spherical coordinates the spatial problem is

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial w}{\partial\rho}\right)+\frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial w}{\partial\phi}\right)+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 w}{\partial\theta^2}+\lambda w=0.$$

Once again we ***separate variables*** with $w(\rho,\theta,\phi)=f(\rho)q(\theta)g(\phi)$ and multiply $\rho^2/(fqg)$, then the spatial equation becomes:

$$\frac{1}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)+\lambda\rho^2=-\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)-\frac{1}{q\sin^2\phi}\frac{d^2q}{d\theta^2}=\mu.$$

The ρ-equation is

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)+\left(\lambda\rho^2-\mu\right)f=0,$$

which is almost **Bessel's equation**.

Spherical Problems

After removing the ρ-equation, the θ and ϕ parts are separated to give:

$$-\frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) - \mu\sin^2\phi = \frac{q''}{q} = -\gamma.$$

The 1^{st} **Sturm-Liouville problem** in θ is:

$$q'' + \gamma q = 0, \qquad \text{with BCs} \quad q(-\pi) = q(\pi) \quad \text{and} \quad q'(-\pi) = q'(\pi),$$

which has ***eigenvalues*** and ***eigenfunctions***

$$\gamma_0 = 0 \quad \text{and} \quad q_0(\theta) = a_0,$$

and

$$\gamma_m = m^2 \quad \text{and} \quad q_m(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta).$$

Associated Legendre Polynomials

The 2^{nd} **Sturm-Liouville problem** in ϕ is:

$$\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\mu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0, \qquad 0 \le \phi \le \pi,$$

with the ***singular BCs*** $g(0)$ and $g(\pi)$ ***bounded***.

This **SL**-problem is related to **associated Legendre polynomials**.

We make the change of variables $x = \cos(\phi)$, $\quad -1 \le x \le 1$, so

$$\frac{d}{d\phi} = \frac{dx}{d\phi}\frac{d}{dx} = -\sin(\phi)\frac{d}{dx}.$$

In the **associated Legendre equation** with the change of variables, the first term is

$$-\sin\phi\frac{d}{dx}\left(-\sin^2\phi\frac{dg}{dx}\right) = \sin\phi\frac{d}{dx}\left((1-\cos^2\phi)\frac{dg}{dx}\right) = \sin\phi\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right).$$

Associated Legendre Polynomials

We divide the **associated Legendre equation** by $\sin(\phi)$ and obtain

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right)+\left(\mu-\frac{m^2}{\sin^2\phi}\right)g=0,$$

which becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right)+\left(\mu-\frac{m^2}{(1-x^2)}\right)g=0.$$

This is a ***Sturm-Liouville problem with regular singular points*** at $x=\pm1$ (or $\phi=0,\pi$) the **poles**.

By writing the equation

$$g''-\frac{2x}{(x+1)(x-1)}g'+\left(\frac{\mu(x^2-1)-m^2}{(x+1)^2(x-1)^2}\right)g=0,$$

it is easy to see that $x=1$ and -1 are ***regular singular points***.

Associated Legendre Polynomials

The **associated Legendre equation** is often written:

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right)+\left(n(n+1)-\frac{m^2}{(1-x^2)}\right)g=0,$$

and its ***linearly independent solutions*** (**associated Legendre functions**) are written:

$$g(x)=c_1P_n^m(x)+c_2Q_n^m(x).$$

It can be shown that when n is not an integer, then both solutions are unbounded at either $x=1$ or $x=-1$.

When n is an integer, then $P_n^m(x)$ is a polynomial, while $Q_n^m(x)$ is unbounded at both $x=1$ and $x=-1$.

Thus, we concentrate our studies on the **associated Legendre polynomials**, $P_n^m(x)$, for our physical problem.

Legendre Polynomials

If $m = 0$ (no θ dependence), **cylindrically symmetric**, **Legendre equation** is given by:

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right) + n(n+1)g = 0.$$

Let $g(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$\frac{d}{dx}\left((1-x^2)\sum_{k=1}^{\infty} a_k k x^{k-1}\right) + n(n+1)\sum_{k=0}^{\infty} a_k x^k = 0.$$

or

$$\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - \sum_{k=0}^{\infty} a_k k(k+1)x^k + n(n+1)\sum_{k=0}^{\infty} a_k x^k = 0.$$

or

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} a_k\big(k(k+1) - n(n+1)\big)x^k = 0.$$

Legendre Polynomials

The power series given by

$$\sum_{k=0}^{\infty}\Big(a_{k+2}(k+2)(k+1) - a_k(k(k+1) - n(n+1))\Big)x^k = 0,$$

has the ***recurrence relation***:

$$a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)}a_k = -\frac{(n-k)(1+n+k)}{(k+2)(k+1)}a_k,$$

where a_0 and a_1 are arbitrary.

It is easy to see by the ***ratio test*** that the series above converges for $|x| < 1$.

When $|x| = \pm 1$, this series diverges unless n is an integer, then one solution of the power series is a ***polynomial***, so **converges**.

Legendre Polynomials

It follows that we can write

$$\begin{aligned} g &= a_0\left(1 - \frac{n(n+1)}{2\cdot 1}x^2 + \frac{(n-2)(n+3)(n+1)n}{4!}x^4 - \dots\right) \\ &\quad + a_1\left(x - \frac{(n-1)(n+2)}{3\cdot 2}x^3 + \frac{(n-3)(n+4)(n-1)(n+2)}{4!}x^5 - \dots\right). \end{aligned}$$

The first **6 Legendre polynomials** are:

$$\begin{array}{ll} n = 0 & P_0(x) = 1, \\ n = 1 & P_1(x) = x, \\ n = 2 & P_2(x) = \frac{1}{2}(3x^2 - 1), \\ n = 3 & P_3(x) = \frac{1}{2}(5x^3 - 3x), \\ n = 4 & P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \\ n = 5 & P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{array}$$

Legendre Polynomials

One method of generating ***Legendre polynomials*** is ***Rodriguez formula***:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n.$$

Since $x = \cos(\phi)$, the first three ***Legendre polynomials*** in ϕ are:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x = \cos(\phi), \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1) = \tfrac{1}{4}\left(3\cos(2\phi) + 1\right). \end{aligned}$$

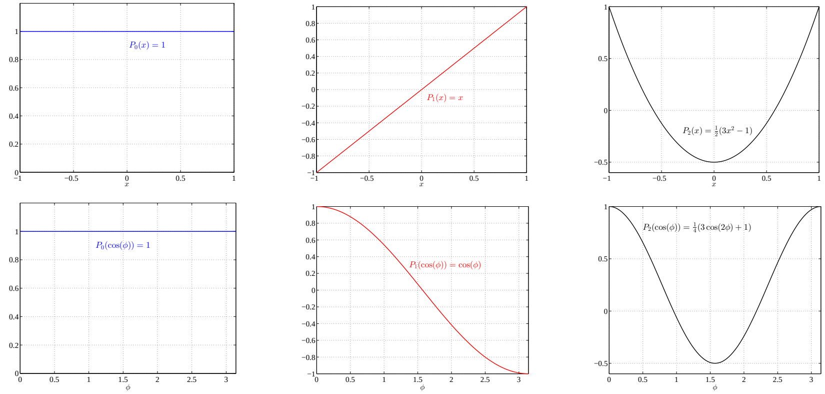
The **orthogonality** has a ***weighting function*** $\sigma(x) = 1$ $(\sigma(\phi) = \sin(\phi))$ and satisfies:

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0, & n \neq m, \\ \frac{2}{2n+1}, & n = m. \end{cases}$$

Uses the recurrence relation and integration by parts.

Legendre Polynomials

Graphs of the first **3 Legendre polynomials**.



Associated Legendre Polynomials

If $m > 0$, then the ***associated Legendre polynomials*** can be found with the formula:

$$g(x) = P_n^m(x) = (x^2 - 1)^{m/2} \frac{d^m}{dx^m} P_n(x),$$

where $n \geq m$ to avoid $g(x) = 0$ and $P_n(x)$ is the ***Legendre polynomial of order*** n.

With these formulas, we have solved for $q(\theta)$ and $g(\phi)$ for the **spherical problem**.

Remains to solve the ***radial*** part of this problem.

Radial Eigenvalue Problem

If the original ***spherical problem*** has ***homogeneous BCs***, $u(a,\theta,\phi,t) = 0$, then the 3^{rd} ***Sturm-Liouville problem*** is

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \Big(\lambda\rho^2 - n(n+1)\Big) f = 0, \qquad f(a) = 0,$$

which is restricted to $n \geq m$ for fixed m.

This is almost ***Bessel's equation***, and it has the solution ***Spherical Bessel's function***:

$$f(\rho) = \rho^{-1/2} J_{n+1/2}\left(\sqrt{\lambda}\rho\right),$$

which are bounded at $\rho = 0$.

The **eigenvalues** satisfy $J_{n+1/2}\left(\sqrt{\lambda}a\right) = 0$, so the k^{th} zero is

$$z_{k,n+1/2} = \sqrt{\lambda_{k,n}}\, a.$$

Radial Eigenvalue Problem

Spherical Bessel functions satisfy

$$x^{-1/2} J_{n+1/2}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx} \right)^n \left(\frac{\sin(x)}{x} \right).$$

The **superposition principle** gives:

$$u(\rho, \theta, \phi, t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} f(\rho) q(\theta) g(\phi) h(t)$$

$$= \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left\{ \begin{array}{c} \cos\left(c\sqrt{\lambda_{k,n}}t\right) \\ \sin\left(c\sqrt{\lambda_{k,n}}t\right) \end{array} \right\} \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{k,n}}\rho\right) \left\{ \begin{array}{c} 1 \\ \cos(m\theta) \\ \sin(m\theta) \end{array} \right\} P_n^m(\cos(\phi)).$$

Laplace in Spherical Cavity

Consider **Laplace's equation in a spherical cavity**:

$$\nabla^2 u = 0, \qquad \text{with} \quad u(a, \theta, \phi) = F(\theta, \phi).$$

In spherical coordinates the spatial problem is

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} = 0.$$

Once again we ***separate variables*** with $u(\rho, \theta, \phi) = f(\rho)q(\theta)g(\phi)$ and multiply $\rho^2/(fqg)$, then the spatial equation becomes:

$$\frac{1}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) = -\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) - \frac{1}{q\sin^2\phi}\frac{d^2 q}{d\theta^2} = \nu.$$

The ρ-equation is

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) - \nu f = 0.$$

Laplace in Spherical Cavity

The ***Sturm-Liouville problems*** are in θ and ϕ.

The θ and ϕ parts are separated to give:

$$-\frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) - \nu\sin^2\phi = \frac{q''}{q} = -\mu.$$

The 1^{st} **Sturm-Liouville problem** in θ is:

$$q'' + \mu q = 0, \qquad \text{with BCs} \quad q(-\pi) = q(\pi) \quad \text{and} \quad q'(-\pi) = q'(\pi),$$

which has ***eigenvalues*** and ***eigenfunctions***

$$\mu_0 = 0 \quad \text{and} \quad q_0(\theta) = a_0,$$

and

$$\mu_m = m^2 \quad \text{and} \quad q_m(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta).$$

Laplace in Spherical Cavity

The 2^{nd} **Sturm-Liouville problem** in ϕ is:

$$\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\nu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0, \qquad 0 \le \phi \le \pi,$$

with the ***singular BCs*** $g(0)$ and $g(\pi)$ ***bounded***.

As seen before, this **SL**-problem is related to **associated Legendre polynomials**.

The solution to this ***eigenvalue problem*** is ***eigenvalues***, $\nu = n(n+1)$ and associated ***eigenfunctions***:

$$g(\phi) = P_n^m(\cos(\phi)).$$

Laplace in Spherical Cavity

The ***radial equation*** satisfies:

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) - n(n+1)f = 0.$$

This is an ***equidimensional*** or ***Euler problem***, so attempt solutions of the form:

$$f(\rho) = \rho^r.$$

The result is:

$$\frac{d}{d\rho}\left(\rho^2 r\rho^{r-1}\right) - n(n+1)\rho^r = 0.$$

This gives

$$\rho^r\Big(r(r+1) - n(n+1)\Big) = \Big(r^2 + r - n(n+1)\Big)\rho^r = 0.$$

Laplace in Spherical Cavity

The above equation is factored to give

$$r^2 + r - n(n+1) = (r-n)(r+n+1) = 0, \quad \text{or} \quad r = n, -(n+1).$$

It follows that

$$f(\rho) = c_1 \rho^n + c_2 \rho^{-(n+1)}.$$

Since the solution is **bounded** at $\rho = 0$, it follows that $c_2 = 0$.

The **superposition principle** gives:

$$\begin{aligned} u(\rho, \theta, \phi) &= \sum_{n=0}^{\infty} A_{0n} \rho^n P_n(\cos(\phi)) \\ &+ \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \rho^n \left(A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta) \right) P_n^m(\cos(\phi)). \end{aligned}$$

Laplace in Spherical Cavity

The **BC** at $\rho = a$ gives:

$$
\begin{aligned}
F(\theta, \phi) &= \sum_{n=0}^{\infty} A_{0n} a^n P_n(\cos(\phi)) \\
&+ \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} a^n \left(A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)\right) P_n^m(\cos(\phi)).
\end{aligned}
$$

Recall that the ***Sturm-Liouville problem*** in ϕ was

$$\frac{d}{d\phi}\left(\sin(\phi)\frac{dg}{d\phi}\right) + \left(n(n+1)\sin(\phi) - \frac{m^2}{\sin(\phi)}\right) g = 0,$$

so the weighting function is $\sigma(\phi) = \sin(\phi)$.

Laplace in Spherical Cavity

The ***Fourier coefficients*** are readily found using ***orthogonality***, so

$$A_{0n} = \frac{\int_{-\pi}^{\pi} \int_0^{\pi} F(\theta, \phi) P_n(\cos(\phi)) \sin(\phi) d\phi \, d\theta}{2\pi a^n \int_0^{\pi} (P_n(\cos(\phi)))^2 \sin(\phi) d\phi \, d\theta},$$

and

$$A_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^{\pi} F(\theta, \phi) \cos(m\theta) P_n^m(\cos(\phi)) \sin(\phi) d\phi \, d\theta}{\pi a^n \int_0^{\pi} (P_n^m(\cos(\phi)))^2 \sin(\phi) d\phi \, d\theta},$$

and

$$B_{mn} = \frac{\int_{-\pi}^{\pi} \int_0^{\pi} F(\theta, \phi) \sin(m\theta) P_n^m(\cos(\phi)) \sin(\phi) d\phi \, d\theta}{\pi a^n \int_0^{\pi} (P_n^m(\cos(\phi)))^2 \sin(\phi) d\phi \, d\theta}.$$