Math 5510 - Partial Differential Equations

Nonhomogeneous Partial Differential Equations

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Outline

Introduction - Nonhomogeneous Problems

Introduction: Separation of Variables requires a linear PDE with homogeneous BCs.

Consider the following ***nonhomogeneous problems***:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - h(u - T_e), \qquad t > 0, \quad 0 < x < L,$$

with **BCs**: $u(0,t) = A$ and $u(L,t) = B$, and **IC**: $u(x,0) = f(x)$.

Begin by solving the ***steady state problem***, $u_E(x)$,

$$ku_E'' - h(u_E - T_e) = 0, \qquad u_E(0) = A \quad \text{and} \quad u_E(L) = B.$$

Equivalently,

$$u_E'' - \frac{h}{k}u_E = -\frac{h}{k}T_e,$$

which is easily seen to have a particular solution, $u_{Ep}(x) = T_e$.

Introduction - Nonhomogeneous Problems

The general solution to the ***steady state problem***, $u_E'' - \frac{h}{k}u_E = -\frac{h}{k}T_e$, is given by

$$u_E(x) = c_1 \cosh\left(\sqrt{\frac{h}{k}}x\right) + c_2 \sinh\left(\sqrt{\frac{h}{k}}x\right) + T_e.$$

The **BCs** give:

$$u_E(0) = c_1 + T_e = A \qquad \text{or} \qquad c_1 = A - T_e,$$

and

$$u_E(L) = (A - T_e)\cosh\left(\sqrt{\frac{h}{k}}L\right) + c_2 \sinh\left(\sqrt{\frac{h}{k}}L\right) + T_e = B.$$

It follows that

$$c_2 = \frac{B - T_e}{\sinh\left(\sqrt{\frac{h}{k}}L\right)} + (T_e - A)\coth\left(\sqrt{\frac{h}{k}}L\right).$$

Introduction - Nonhomogeneous Problems

Now let $v(x,t) = u(x,t) - u_E(x)$, so $u = v + u_E$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + ku_E'' - h(v + u_E - T_e).$$

However, $ku_E'' - h(u_E - T_e) = 0$, so the above PDE becomes the ***homogeneous PDE*** for $v(x,t)$

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} - hv,$$

with the ***homogeneous BCs***: $v(0,t) = 0$ and $v(L,t) = 0$, and the ***IC***: $v(x,0) = f(x) - u_E(x)$.

Our previous techniques of ***separation of variables*** applies to this problem, so let $v(x,t) = \phi(x)g(t)$, and

$$\phi g' = kg\phi'' - h\phi g \qquad \text{or} \qquad \frac{g' + hg}{kg} = \frac{\phi''}{\phi} = -\lambda.$$

Introduction - Nonhomogeneous Problems

The **Sturm-Liouville problem** is

$$\phi'' + \lambda\phi = 0, \qquad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

As we have often seen before, this has ***eigenvalues*** and ***eigenfunctions***:

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad \text{and} \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

The solution to the t-equation is

$$g(t) = ce^{-(h+\lambda k)t}.$$

By the **superposition principle**, the solution becomes:

$$v(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(h+\frac{kn^2\pi^2}{L^2}\right)t} \sin\left(\frac{n\pi x}{L}\right).$$

Introduction - Nonhomogeneous Problems

We apply the **IC**, so

$$v(x,0) = f(x) - u_E(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

which has the Fourier coefficients:

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x)) \sin\left(\frac{n\pi x}{L}\right) dx.$$

The solution to the original ***nonhomogeneous problem*** is

$$u(x,t) = v(x,t) + u_E(x),$$

where $u_E(x)$ is the solution of the ***steady-state*** problem and $v(x,t)$ is the solution above to the ***homogeneous PDE***.

Time-dependent Nonhomogeneous Terms

Consider the ***time-dependent nonhomogeneous PDE***:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

with ***time-dependent BCs***:

$$u(0,t) = A(t) \qquad \text{and} \qquad u(L,t) = B(t),$$

and ***IC***: $\quad u(x,0) = f(x).$

Create a related problem with ***homogeneous BCs***.

Consider any ***reference temperature distribution***, $r(x,t)$, where ***simpler is better***, such that

$$r(0,t) = A(t) \qquad \text{and} \qquad r(L,t) = B(t).$$

For example,

$$r(x,t) = A(t) + \frac{x}{L}(B(t) - A(t)).$$

Time-dependent Nonhomogeneous Terms

Take $v(x,t) = u(x,t) - r(x,t)$, then the PDE becomes:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \left(Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}\right) \equiv k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t)$$

with ***homogeneous BCs***:

$$v(0,t) = 0 \qquad \text{and} \qquad v(L,t) = 0,$$

and ***IC***: $\quad v(x,0) = f(x) - r(x,0).$

Note: Our choice of $r(x,t)$ being linear in x gives $r_{xx} = 0$, simplifying the PDE above and $\bar{Q}(x,t)$, in particular.

Method of Eigenfunction Expansion

The use of a ***reference function*** readily converts nonhomogeneous BCs to one with homogeneous BCs, so what about nonhomgeneities in the PDE?

Consider the problem:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t),$$

with ***homogeneous BCs***:

$$v(0,t) = 0 \qquad \text{and} \qquad v(L,t) = 0,$$

and ***IC***: $\quad v(x,0) = g(x)$.

The ***related homogeneous problem*** is:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with ***homogeneous BCs***:

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

Method of Eigenfunction Expansion

The problem, $u_t = ku_{xx}$, with $u(0,t) = 0$ and $u(L,t) = 0$, has been shown to have ***eigenvalues*** and ***eigenfunctions***:

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

To solve the ***nonhomogeneous problem*** in $v(x,t)$, we attempt a solution of the form:

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

where $\phi_n(x)$ are any ***eigenfunctions*** of the related ***homogeneous problem*** (often different BCs).

Method of Eigenfunction Expansion

The **IC** is

$$v(x,0) = g(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x),$$

so

$$a_n(0) = \frac{\int_0^L g(x)\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx}.$$

This can be easily generalized to ***Sturm-Liouville problems*** with different weighting functions.

If v and $\frac{\partial v}{\partial x}$ are continuous and $v(x,t)$ solves the same homogeneous BCs as $\phi_n(x)$, then term-by-term differentiation can be justified.

We showed this for the Fourier sine and cosine series, but general Sturm-Liouville problems have the same properties and related theorems.

Method of Eigenfunction Expansion

With $v(x,t)$ given by:

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

the term-by-term differentiation gives:

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} \frac{d\,a_n(t)}{dt}\phi_n(x),$$

and

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} a_n(t)\frac{d^2\phi_n(x)}{d^2x} = -\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x).$$

This leaves us with the ***system of linear ODEs***:

$$\sum_{n=1}^{\infty}\left[\frac{d\,a_n(t)}{dt} + \lambda_n k a_n(t)\right]\phi_n(x) = \bar{Q}(x,t),$$

where our previous Fourier series for the ICs gave the values for $a_n(0)$.

Method of Eigenfunction Expansion

The left hand side of the equation

$$\sum_{n=1}^{\infty}\left[\frac{d\,a_n(t)}{dt}+\lambda_n k a_n(t)\right]\phi_n(x)=\bar{Q}(x,t),$$

gives the Fourier expansion of $\bar{Q}(x,t)$.

Assuming that

$$\bar{Q}(x,t)=\sum_{n=1}^{\infty}\bar{q}_n(t)\phi_n(x),$$

then the ***orthogonality*** of the eigenfunctions gives the system of ODEs:

$$\frac{d\,a_n(t)}{dt}+\lambda_n k a_n(t)=\bar{q}_n(t)=\frac{\int_0^L \bar{Q}(x,t)\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx},\quad n=1,2,...$$

This system of ODEs is solved with the ***variation of parameters method***, giving

$$a_n(t)=a_n(0)e^{-\lambda_n kt}+e^{-\lambda_n kt}\int_0^t \bar{q}_n(s)e^{\lambda_n ks}ds.$$

The nonhomogeneous solution becomes $v(x,t)=\sum_{n=1}^{\infty}a_n(t)\phi_n(x)$.

Example for Eigenfunction Expansion

Consider the ***nonhomogeneous PDE*** given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}\sin(3x), \qquad 0 < x < \pi, \quad t > 0.$$

Assume BCs given by $u(0,t) = 0$ and $u(\pi,t) = 1$ and IC given by $u(x,0) = f(x)$.

We create a problem with homogeneous BCs by using a simple ***reference function***, $r(x) = x/\pi$, so take

$$v(x,t) = u(x,t) - \frac{x}{\pi}.$$

The new homogeneous problem for $v(x,t)$ becomes:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + e^{-t}\sin(3x),$$

with BCs and IC:

$$v(0,t) = 0, \qquad v(\pi,t) = 0, \quad \text{and} \quad v(x,0) = f(x) - \frac{x}{\pi}.$$

Example for Eigenfunction Expansion

The problem $v_t = v_{xx}$ with BC $v(0,t) = 0 = v(\pi,t)$ has ***eigenvalues***, $\lambda_n = n^2$, with associated ***eigenfunctions***, $\phi_n(x) = \sin(nx)$.

Thus, we use the ***eigenfunction expansion***:

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx).$$

We insert this expansion into the ***nonhomogeneous problem***:

$$\sum_{n=1}^{\infty} \frac{d\,a_n(t)}{dt}\sin(nx) = -n^2\sum_{n=1}^{\infty} a_n(t)\sin(nx) + e^{-t}\sin(3x),$$

which can be written:

$$\sum_{n=1}^{\infty}\left(\frac{d\,a_n(t)}{dt} + n^2 a_n(t)\right)\sin(nx) = e^{-t}\sin(3x).$$

Example for Eigenfunction Expansion

The Fourier coefficients are found by multiplying by $\sin(mx)$ and integrating from $x = 0$ to $x = \pi$, giving

$$\frac{d\,a_n}{dt} + n^2 a_n = \begin{cases} 0, & n \neq 3, \\ e^{-t}, & n = 3. \end{cases}$$

The solution to these equations are

$$a_n(t) = \begin{cases} a_n(0)e^{-n^2 t}, & n \neq 3, \\ \frac{1}{8}e^{-t} + \left(a_3(0) - \frac{1}{8}\right)e^{-9t}, & n = 3. \end{cases},$$

where

$$a_n(0) = \frac{2}{\pi}\int_0^{\pi}\left(f(x) - \frac{x}{\pi}\right)\sin(nx)\,dx.$$

The solution satisfies:

$$u(x,t) = v(x,t) + \frac{x}{\pi}.$$

Eigenfunction Expansion and Green's Formula

Consider the **PDE**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

with **BCs** and **IC**:

$$u(0,t) = A(t), \qquad u(L,t) = B(t), \qquad u(x,0) = f(x).$$

The related ***homogeneous BVP*** is

$$\frac{d^2\phi_n}{dx^2} + \lambda_n\phi_n = 0, \qquad \phi_n(0) = 0 = \phi_n(L),$$

which has ***eigenvalues*** and corresponding ***eigenfunctions***:

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Eigenfunction Expansion and Green's Formula

Expand the $u(x,t)$ in term of the ***eigenfunctions***:

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\phi_n(x).$$

1. This expansion fails at the boundaries, since $\phi_n(x)$ are homogeneous, while $u(x,t)$ is not.
2. We can **NOT** differentiate w.r.t. x because of the different BCs for ϕ_n and u.
3. However, term-by-term differentiation by t is valid.

We write

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{db_n}{dt}\phi_n(x).$$

Eigenfunction Expansion and Green's Formula

It follows that

$$\sum_{n=1}^{\infty} \frac{db_n}{dt}\phi_n(x) = k\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

so

$$\frac{db_n}{dt} = \frac{\int_0^L \left[k\frac{\partial^2 u}{\partial x^2} + Q(x,t)\right]\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}.$$

If $Q(x,t)$ has a ***generalized Fourier expansion***

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x), \quad \text{with} \quad q_n(t) = \frac{\int_0^L Q(x,t)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx},$$

then

$$\frac{db_n}{dt} = q_n(t) + \frac{\int_0^L k\frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}.$$

Eigenfunction Expansion and Green's Formula

Recall that when L is any ***Sturm-Liouville operator*** with

$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

we had **Green's formula**

$$\int_0^L [uL(v) - vL(u)]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_0^L.$$

In our example, we have the operator

$$L = \frac{\partial^2}{\partial x^2} \qquad \text{with} \quad p(x) = 1.$$

We can use partial derivatives in **Green's formula** with t fixed.

Eigenfunction Expansion and Green's Formula

Let $v(x) = \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, so $\frac{dv}{dx} = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$.

By **Green's formula**,

$$\begin{aligned}\int_0^L \phi_n(x) L(u)\,dx &= \int_0^L uL(v)\,dx + \left(v\frac{\partial u}{\partial x} - u\frac{dv}{dx}\right)\Bigg|_0^L, \\ &= -\lambda_n \int_0^L u\phi_n\,dx - \frac{n\pi}{L}\left[u(L,t)\cos(n\pi) - u(0,t)\right], \\ &= -\lambda_n \int_0^L u\phi_n\,dx - \frac{n\pi}{L}\left[B(t)(-1)^n - A(t)\right].\end{aligned}$$

However, $b_n(t)$ are the ***generalized Fourier coefficients*** of $u(x,t)$, so

$$b_n(t) = \frac{\int_0^L u\phi_n\,dx}{\int_0^L \phi_n^2\,dx}.$$

Eigenfunction Expansion and Green's Formula

The information above is substituted into the DE for $b_n(t)$ and

$$\frac{db_n(t)}{dt} + k\lambda_n b_n = q_n(t) + \frac{kn\pi}{L\int_0^L \phi_n^2\,dx}\left[B(t)(-1)^n - A(t)\right].$$

The ICs give

$$f(x) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x), \qquad \text{so} \quad b_n(0) = \frac{\int_0^L f(x)\phi_n(x)\,dx}{L\int_0^L \phi_n^2\,dx}.$$

The above 1^{st} ***order differential equation*** in $b_n(t)$ with its IC has a unique solution, solving the PDE in $u(x,t)$.

If the PDE in $u(x,t)$ has homogeneous BCs, then the ***eigenfunction expansion*** solution converges much faster than if the BCs are nonhomogeneous.

Green's Functions

Consider the **Heat Equation**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad 0 < x < L,$$

with BCs and IC:

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad u(x,0) = g(x).$$

The solution from before is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t},$$

where the initial condition gives the Fourier coefficients

$$g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \qquad \text{so} \quad a_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

Green's Functions

We want to examine more closely the effect of the IC $g(x)$.

Introduce a dummy variable x_0 and substitute in the Fourier coefficient:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L g(x_0) \sin\left(\tfrac{n\pi x_0}{L}\right) dx_0 \right) \sin\left(\tfrac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}.$$

Interchange the summation and integration to obtain:

$$u(x,t) = \int_0^L g(x_0) \left(\sum_{n=1}^{\infty} \frac{2}{L} \sin\left(\tfrac{n\pi x_0}{L}\right) \sin\left(\tfrac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t} \right) dx_0.$$

The quantity in the parentheses is the **influence function** for the initial condition.

It expresses the contribution of the temperature at x and t due to the initial temperature at x_0. The solution $u(x,t)$ is the integral over all influences from all the positions of the IC.

Green's Functions

If we extend the previous analysis to the **PDE**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

where the BCs are the same homogeneous ones and $u(x,0) = g(x)$.

From our ***eigenfunction expansion*** technique, we write:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

This is differentiated term-by-term because of the homogeneous BCs, so

$$\frac{da_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 a_n = q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\sin\left(\frac{n\pi x}{L}\right)\,dx,$$

where

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi x}{L}\right).$$

Green's Functions

The ODE for $a_n(t)$ has the solution:

$$a_n(t) = a_n(0)e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t} \int_0^t q_n(t_0) e^{k(n\pi/L)^2 t_0}\, dt_0,$$

where $u(x,0) = g(x)$, so

$$g(x) = \sum_{n=1}^{\infty} a_n(0) \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad a_n(0) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

The Fourier coefficients are eliminated to produce:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left[\left(\frac{2}{L} \int_0^L g(x_0) \sin\left(\frac{n\pi x_0}{L}\right)\, dx_0 \right) e^{-k(n\pi/L)^2 t} \right. \\ &+ \left. e^{-k(n\pi/L)^2 t} \int_0^t \left(\frac{2}{L}\int_0^L Q(x_0,t_0)\sin\left(\frac{n\pi x_0}{L}\right)\, dx_0 \right) e^{k(n\pi/L)^2 t_0}\, dt_0 \right] \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

Green's Functions

Interchanging the order of summation and integration gives:

$$\begin{aligned} u(x,t) &= \int_0^L g(x_0) \left(\sum_{n=1}^{\infty} \frac{2}{L} \sin\left(\frac{n\pi x_0}{L}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t} \right) dx_0 \\ &+ \int_0^L \int_0^t Q(x_0,t_0) \left(\sum_{n=1}^{\infty} \frac{2}{L} \sin\left(\frac{n\pi x_0}{L}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 (t-t_0)} \right) dt_0\, dx_0. \end{aligned}$$

Define the **Green's function**, $G(x,t;x_0,t_0)$,

$$G(x,t;x_0,t_0) = \sum_{n=1}^{\infty} \frac{2}{L} \sin\left(\frac{n\pi x_0}{L}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 (t-t_0)}$$

The solution can be written:

$$\begin{aligned} u(x,t) &= \int_0^L g(x_0) G(x,t;x_0,0)\, dx_0 \\ &\quad + \int_0^L \int_0^t Q(x_0,t_0) G(x,t;x_0,t_0)\, dt_0\, dx_0. \end{aligned}$$

Green's Functions

- The **Green's function**, $G(x, t; x_0, 0)$, expresses the ***influence*** of the initial temperature at position x and time t
- The **Green's function**, $G(x, t; x_0, t_0)$, gives the ***influence*** on position x at time t of the forcing term, $Q(x_0, t_0)$
- The **Green's function** depends only on the **elapsed time**, $t - t_0$,

$$G(x, t; x_0, t_0) = G(x, t - t_0; x_0, 0).$$

- The **Heat equation** is independent of time, so thermal properties are not changing.
- The most recent time events are most important.
- The series converges more slowly for small t, while $G(x, t; x_0, t_0)$ more accurately describes long time behavior.
- The solution $u(x, t)$ given with the **Green's function** gives the ***influences*** over all x_0 and past time $0 < t_0 < t$.
- This gives the **causality principle** where the temperature depends on the ***thermal sources*** acting before the current time, t.