Math 5510 - Partial Differential Equations

Fourier Transforms for PDEs

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Outline

Infinite Domain Problems

Heat Equation: Consider the PDE on an infinite domain:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad -\infty < x < \infty,$$

with IC

$$u(x,0) = f(x).$$

Additionally, the IC satisfies:

$$\lim_{x\to\pm\infty} f(x) = 0.$$

The BCs are

$$\lim_{x\to\pm\infty} u(x,t) = 0.$$

Infinite Domain Problems

As before, we use ***separation of variables***:

$$u(x,t) = \phi(x)g(t).$$

From the PDE, we obtain

$$\phi(x)g'(t) = k\phi''(x)g(t), \qquad \text{or} \qquad \frac{g'}{kg} = \frac{\phi''}{\phi} = -\lambda.$$

The ***eigenvalue problem*** becomes:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad |\phi(\pm\infty)| < \infty.$$

This has bounded solutions for $\lambda \geq 0$. In particular, if $\lambda = \omega^2$, then

$$\phi(x) = c_1\cos(\omega x) + c_2\sin(\omega x).$$

Infinite Domain Problems

From before, any $\lambda \geq 0$ solves the ***eigenvalue problem***,

so we obtain a ***continuous spectrum*** for $\lambda \geq 0$.

The solution to the t-dependent equation is:

$$g(t) = e^{-k\omega^2 t}.$$

Superposition principle: Since the eigenvalues form a ***continuous spectrum***, the ***superposition principle*** requires integration over the ***continuous spectrum***, rather than an infinite sum.

The solution becomes:

$$u(x,t) = \int_0^\infty \left[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x) \right] e^{-k\omega^2 t}\, d\omega.$$

Infinite Domain Problems

It remains to show this satisfies the **IC**, so

$$u(x,0) = f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\, d\omega.$$

It remains to show there exist $A(\omega)$ and $B(\omega)$, which are valid for most functions, $f(x)$.

Complex exponentials: Recall that **Euler's formula** gives:

$$\cos(\omega x) = \frac{e^{i\omega x} + e^{-i\omega x}}{2} \qquad \text{and} \qquad \sin(\omega x) = \frac{e^{i\omega x} - e^{-i\omega x}}{2i},$$

so complex solutions are linear combinations of complex exponentials.

An alternate way to write the solution is:

$$u(x,t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t}\, d\omega.$$

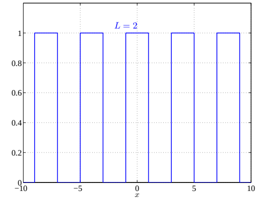
Convention uses $e^{-i\omega x}$ with $|\omega|$ being the wave number.

Fourier Integral

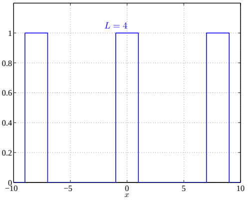
Example: Consider the periodic function:

$$f_L(x) = \begin{cases} 0, & -L < x < -1, \\ 1, & -1 < x < 1, \\ 0, & 1 < x < L, \end{cases}$$

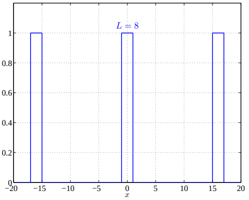
where $f_L(x + 2Ln) = f_L(x)$ for all integers n, creating a $2L$-periodic function.



$f_2(x)$



$f_4(x)$



$f_8(x)$

Fourier Integral

Example: The limiting case is given by:

$$f(x) = \begin{cases} 1, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The functions above are even, so the Fourier series contains only cosine terms, so

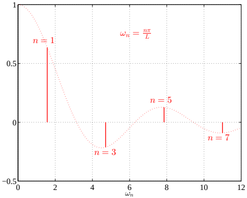
$$f_L(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

The Fourier coefficients are:

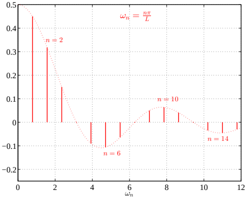
$$a_0 = \frac{1}{2L}\int_{-1}^{1} 1\, dx = \frac{1}{L} \quad \text{and} \quad a_n = \frac{1}{L}\int_{-1}^{1} \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{n\pi}\sin\left(\frac{n\pi}{L}\right).$$

Fourier Integral

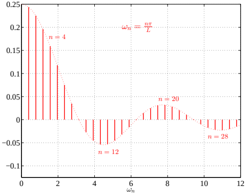
Fourier coefficients: The sequence of Fourier coefficients is called the ***amplitude spectrum*** of $f_L(x)$ because $|a_n|$ is the maximum amplitude of $a_n \cos\left(\frac{n\pi x}{L}\right)$, where $a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{L}\right) = \frac{2}{n\pi}\sin(\omega_n)$.



$L = 2$



$L = 4$



$L = 8$

The ***amplitude spectrum*** becomes denser as L increases.

Thus, the discrete system approaches the continuous system.

Fourier Series

Fourier Series: The Fourier series for $f_L(x)$ is

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(\omega_n x) + b_n \sin(\omega_n x)\right), \qquad \omega_n = \tfrac{n\pi}{L}.$$

With the Fourier coefficient formulas,

$$\begin{aligned} f_L(x) &= \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos(\omega_n x)\int_{-L}^{L} f_L(v)\cos(\omega_n v)\,dv \\ &\quad + \sin(\omega_n x)\int_{-L}^{L} f_L(v)\sin(\omega_n v)\,dv\Big]. \end{aligned}$$

Define $\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$, so $\frac{1}{L} = \frac{\Delta\omega}{\pi}$.

Fourier Series to Fourier Integral

With the information that the normalization $\frac{1}{L} = \frac{\Delta\omega}{\pi}$ and for all finite L, we have

$$\begin{aligned} f_L(x) &= \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos(\omega_n x)\Delta\omega \int_{-L}^{L} f_L(v)\cos(\omega_n v)\,dv \right. \\ &\qquad \left. + \sin(\omega_n x)\Delta\omega \int_{-L}^{L} f_L(v)\sin(\omega_n v)\,dv\right]. \end{aligned}$$

Let $L \to \infty$, then

$$f(x) = \lim_{L\to\infty} f_L(x).$$

It is plausible (assuming $f(x)$ ***absolutely integrable***) that

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos(\omega x)\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv + \sin(\omega x)\int_{-\infty}^{\infty} f(v)\sin(\omega v)\,dv\right] d\omega.$$

Definition (Absolutely Integrable)

A function $f(x)$ is ***absolutely integrable*** if the limits exists for

$$\lim_{a\to-\infty}\int_a^0 |f(x)|\,dx + \lim_{b\to\infty}\int_0^b |f(x)|\,dx.$$

Fourier Integral

In the limiting case, the ***Fourier series*** naturally transformed to the ***Fourier Integral***:

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\cos(\omega x)\int_{-\infty}^\infty f(v)\cos(\omega v)\,dv + \sin(\omega x)\int_{-\infty}^\infty f(v)\sin(\omega v)\,dv\right]\,d\omega,$$

$$f(x) = \int_0^\infty \left[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right]\,d\omega, \qquad (1)$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos(\omega v)\,dv \qquad \text{and} \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin(\omega v)\,dv.$$

Theorem (Fourier Integral)

If $f(x)$ is piecewise smooth in every finite interval and $f(x)$ is absolutely integrable, then $f(x)$ can be represented by a Fourier integral (1). At a point of discontinuity the value of the Fourier integral equals the midpoint of the left and right hand limits of $f(x)$ at that point.

Fourier Integral - Example

Example: Consider the example:

$$f(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

The ***Fourier integral*** representation is:

$$f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\, d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)\, dv = \frac{1}{\pi}\int_{-1}^{1}\cos(\omega v)\, dv = \frac{2\sin(\omega)}{\pi\omega}$$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)\, dv = \frac{1}{\pi}\int_{-1}^{1}\sin(\omega v)\, dv = 0.$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos(\omega x)\sin(\omega)}{\omega}\, d\omega = \begin{cases} 1, & |x| < 1, \\ \frac{1}{2}, & |x| = 1, \\ 0, & |x| > 1. \end{cases}$$

Fourier Transform Pair

We omitted the ***complex Fourier series*** earlier, but it satifies:

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}.$$

The function $f(x)$ is $2L$-periodic with

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx.$$

For periodic functions, $-L < x < L$, the allowable ***wave numbers*** ω are

$$\omega = \frac{n\pi}{L} = 2\pi\frac{n}{2L},$$

where the ***wave lengths*** are $\frac{2L}{n}$, which are integral partitions of the region length $2L$.

The distance between successive values of the ***wave number*** is:

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Fourier Transform

With the frequency $\omega = \frac{n\pi}{L}$ and the normalization $\frac{1}{2L} = \frac{\Delta\omega}{2\pi}$ follows that the ***complex Fourier series*** can be written:

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \left(\int_{-L}^{L} f(s) e^{i\omega s}\, ds \right) e^{-i\omega x}.$$

Fourier Transform is the limiting form as $L \to \infty$.

The values ω are the square root of the ***eigenvalues***, and as $L \to \infty$, the ***eigenvalues*** get closer together, approaching a continuum.

Definition (Fourier Integral Identity)

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(s) e^{i\omega s}\, ds \right] e^{-i\omega x}\, d\omega.$$

Fourier Transform

Definition (Fourier Transform)

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{i\omega s}\, ds.$$

It follows that

$$\frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega.$$

Note: Different authors do different things with the $\frac{1}{2\pi}$ factor, so watch the definitions carefully.

If $f(x)$ is continuous, then the ***Fourier integral representation of*** $f(x)$ is

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega.$$

Fourier Transform

Definition (Fourier Transform Pair)

If $f(x)$ is continuous, then the ***Fourier integral representation of*** $f(x)$ is

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega,$$

where

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{i\omega s}\, ds$$

The two equations above are called the ***Fourier Transform pair***.

This relationship shows that $f(x)$ is composed of ***waves*** $e^{-i\omega x}$ for all wave numbers and wave lengths.

The ***Fourier Transform pair*** is valid provided $f(x)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

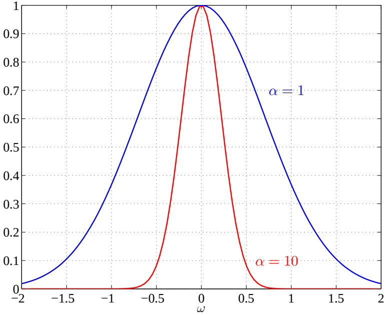
Generally, we also want $f(x)$ to also be piecewise smooth, but this condition can be relaxed.

Inverse Fourier Transform

The ***Gaussian function*** often arises from the ***diffusion operator***:

$$G(\omega) = e^{-\alpha\omega^2}.$$

The function whose ***Fourier transform*** is $G(\omega)$:



$$g(x) = \int_{-\infty}^{\infty} G(\omega) e^{-i\omega x}\, d\omega = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega$$

or

$$g(x) = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha}.$$

Fourier Transform

It follows that the inverse of a ***Gaussian*** is itself a ***Gaussian***:

Fourier Transform Table

$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$	$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
$f(x) = e^{-\beta x^2}$	$F(\omega) = \frac{1}{\sqrt{4\pi\beta}} e^{-\omega^2/4\beta}$
$f(x) = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha}$	$F(\omega) = e^{-\alpha\omega^2}$

Derivation: Consider:

$$g(x) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega.$$

The derivative is:

$$\frac{dg}{dx} = \int_{-\infty}^{\infty} -i\omega e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega.$$

Fourier Transform

Integration by parts with vanishing at the endpoints:

$$\frac{dg}{dx} = -\frac{i}{2\alpha}\int_{-\infty}^{\infty}\frac{d}{d\omega}\left(e^{-\alpha\omega^2}\right)e^{-i\omega x}\,d\omega = -\frac{x}{2\alpha}\int_{-\infty}^{\infty}e^{-\alpha\omega^2}e^{-i\omega x}\,d\omega = -\frac{x}{2\alpha}g(x).$$

The solution of this ODE is

$$g(x) = g(0)e^{-x^2/4\alpha}, \qquad \text{where} \quad g(0) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2}\,d\omega.$$

Let $z = \sqrt{\alpha}\omega$ (or $dz = \sqrt{\alpha}d\omega$), so

$$g(0) = \frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\frac{\pi}{\alpha}}.$$

Note: There are two ways to solve this (one involves complex variables):

$$\left(\int_{-\infty}^{\infty} e^{-z^2}\,dz\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta = \pi.$$