Theorem

A linear system of equations has exactly one of the following:

i No solution, or

ii Exactly one solution, or

iii Infinitely many solutions.

Terms: A system is
consistent if it has at least one solution (cases ii and iii), and
inconsistent if is has no solutions (case i).

Two critical questions about any linear system are: (1) Does it have a solution? (existence), and (2) If it has a solution, is there only one? (uniqueness)

Matrices

Definition: A matrix is a rectangular array of numbers. It's **size** (a.k.a. dimension/order) is $m \times n$ (read "m by n") where m is the number of rows and n is the number of columns the matrix has.

Examples:

$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 13 & -4 \\ 12 & -3 & 2 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 3 & -5 \end{bmatrix}$$

3 x 4   3 x 2

Linear System & Matrices

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Before we start to set up these matrices, we write our system in the form shown above. Note that all variables are on the left side, and like variables have the same order in each equation (they are aligned vertically).

Linear System: Coefficient Matrix

The **coefficient** matrix has one row for each equation and one column for each variable. The entries are the coefficients of the variables in our system.

m = # of equations

n = # of variables

Example:
$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Here $m=3$, $n=3$ the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Linear System: Augmented Matrix

The **augmented** matrix has one row for each equation, one column for each variable, and one extra, right most column. The entries in the first columns match the coefficient matrix, and the right most column has the numbers from the right hand side of each equation.

Example:
$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ 2x_1 \phantom{{}+2x_2} + x_3 &= 7 \\ x_1 + x_2 + x_3 &= 6 \end{aligned}$$

m = # equations

n = # of variables + 1

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Legitimate Operations for Solving a System

We can perform three basic operation **without** changing the solution set of a system. These are

- swap the order of any two equations **(swap)**,
- multiply an equation by any nonzero constant **(scale)**, and
- replace an equation with the sum of itself and a nonzero multiple of any other equation **(replace)**.

We'll try to solve a system by using these operations to *eliminate* variables from equations.

Some Operation Notation

Notation

- Swap equations i and j:
$$E_i \leftrightarrow E_j$$
- Scale equation i by k:
$$kE_i \rightarrow E_i$$
- Replace equation j with the sum of itself and k times equation i:
$$kE_i + E_j \rightarrow E_j$$

Solve the following system of equations by *elimination*.

Keep tabs on the augmented matrix at each step.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Eliminate x_1 from E_2 and E_3

$-2E_1 + E_2 \to E_2$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & -4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$-E_1 + E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -4x_2 + 3x_3 &= 15 \\ -x_2 + 2x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 0 & -1 & 2 & 10 \end{bmatrix}$$

lets swap E_2 and E_3

$E_2 \longleftrightarrow E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -4x_2 + 3x_3 &= 15 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & -4 & 3 & 15 \end{bmatrix}$$

Eliminate x_2 from E_3

$$-4E_2 + E_3 \to E_3$$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -5x_3 &= -25 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & -5 & -25 \end{bmatrix}$$

Scale E_3 $-\frac{1}{5}E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Scale E_2 $\quad -E_2 \to E_2$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ x_2 - 2x_3 &= -10 \\ x_3 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

From the system, we see that $x_3 = 5$.

Substituting this into E_2

$$x_2 = -10 + 2x_3 = -10 + 2(5) = 0$$

Substituting x_2 and x_3 into the first equation

$$x_1 + 2x_2 - x_3 = -4$$

$$\Rightarrow \quad x_1 = -4 - 2x_2 + x_3 = -4 - 2(0) + 5 = 1$$

The system is consistent with exactly one solution

$$x_1 = 1$$

$$x_2 = 0$$

$$x_3 = 5$$

We could write this solution as $(1, 0, 5)$.

Elementary Row Operations: On a Matrix

If any sequence of the following operations are performed on a matrix, the resulting matrix is **row equivalent**.

i Interchange any two rows (**row swap**).
ii Multiply a row by any nonzero constant (**scaling**).
iii Replace a row with the sum of itself and a multiple of another row (**replacement**).

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems have the same solution set. (i.e. The systems are equivalent!)