Section 1.2: Row Reduction and Echelon Forms

We defined the following **Elementary Row Operations**

i Interchange any two rows (**row swap**).
ii Multiply a row by any nonzero constant (**scaling**).
iii Replace a row with the sum of itself and a multiple of another row (**replacement**).

We said that if a sequence of these operations transforms a matrix, the result is called **Row Equivalent.**

Theorem: If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

Why is Row Equivalence Interesting?

The following matrices are row equivalent.

$$\begin{bmatrix} 1 & -1 & -1 & 7 \\ 1 & 2 & 2 & 1 \\ \frac{1}{2} & \frac{7}{2} & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Echelon Forms

ref

Definition: A matrix is in **echelon form** (a.k.a. **row echelon form**) if the following properties hold

i Any row of all zeros are at the bottom.

ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.

iii All entries below a leading entry are zeros.

Is	Is Not
$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Reduced Echelon Form

rref

Definition: A matrix is in **reduced echelon form** (a.k.a. **reduced row echelon form**) if it is in echelon form and the following additional properties hold

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

Is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Is Not

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row Operations to Find ref's and rref's

We'll use the following notation for the three types of elementary row operations. We let R_i represent the i^{th} row in a matrix.

- Swap rows i and j:
$$R_i \leftrightarrow R_j$$
- Scale row i by k:
$$kR_i \rightarrow R_i$$
- Replace row j with the sum of itself and k times row i:
$$kR_i + R_j \rightarrow R_j$$

Example (finding ref's and rref's)

Find an echelon form for the following matrix using elementary row operations.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

We'll work top down from left to right.

Clear out the column for the 1st leading entry. Get 0 where the 4 is.

Scratch work

$$\begin{matrix} -4 & -2 & -6 \\ 4 & 3 & 6 \end{matrix}$$

$$-2R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

Move to row 2. We need 0 where the 3 is in R_3.

$-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Scratch work

$$\begin{matrix} 0 & -3 & 0 \\ 0 & 3 & 2 \end{matrix}$$

This is an ref for the matrix.

Find the reduced echelon form for the following matrix.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

we'll start from the echelon form

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

will work right to left and bottom up.

Need leading 1 in R_3.

$\frac{1}{2}R_3 \to R_3$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Need zeros above the right most leading 1

$-3R_3 + R_1 \to R_1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Need zero above the middle leading 1

$-R_2 + R_1 \to R_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Need the 1st leading entry to be a 1

$\frac{1}{2}R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is an rref.

Theorem: The reduced row echelon form of a matrix is unique.

This allows the following unambiguous definition:

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Identify the pivot position and columns given...

$$A \qquad\qquad \text{rref of } A$$

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot positions are row 1 colum 1, row 2 column 2, and row 3 column 4.

Pivot columns are columns 1, 2, and 4.

Complete Row Reduction isn't needed to find Pivots

Find the pivot positions and pivot columns of the matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

The leading entries in any ref tell you where the pivot positions are.

pivot positions are row 1 column 1 and row 2 column 2. Pivot columns are columns 1 and 2.

This matrix has an ref and rref

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{respectively.}$$

Row Reduction Algorithm

To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$$

$(R_1 \leftrightarrow R_3)$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$-R_1 + R_2 \to R_2$

Step 1: The left most column is a pivot column. The top position is a pivot position.

Step 2: Get a nonzero entry in the top left position by row swapping if needed.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$-R_1 + R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$\frac{1}{2} R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 3: Use row operations to get zeros in all entries below the pivot.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

will finish this out next time

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3.