Math 5510/Math 4510 - Partial Differential Equations: Separation of Variables

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Outline

Homogeneous

Heat Equation: Assume a uniform rod of length L, so that the diffusivity, specific heat, and density do not vary in x

The general **heat equation** satisfies the partial differential equation (PDE):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}, \qquad t > 0, \quad 0 < x < L,$$

with initial conditions (ICs):

$$u(x,0) = f(x), \qquad 0 < x < L,$$

and **Dirichlet boundary conditions** (BCs):

$$u(0,t) = T_1(t) \quad \text{and} \quad u(L,t) = T_2(t), \quad t > 0.$$

If $Q(x,t) \equiv 0$, then the PDE is ***homogeneous***.

If $T_1(t) \equiv T_2(t) \equiv 0$, then the BCs are ***homogeneous***.

Linearity

Definition (Linearity)

An operator $\mathcal{L}$ is linear if and only if

$$\mathcal{L}[c_1 u_1 + c_2 u_2] = c_1 \mathcal{L}[u_1] + c_2 \mathcal{L}[u_2]$$

for any two functions u_1 and u_2 and constants c_1 and c_2.

Define the **Heat Operator**

$$\frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} = \mathcal{L}.$$

Principle of Superposition

The following shows ***linearity*** of the **Heat Operator**:

$$\begin{aligned}\mathcal{L}[c_1u_1 + c_2u_2] &= \left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)(c_1u_1 + c_2u_2) \\ &= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} - kc_2\frac{\partial^2 u_2}{\partial x^2} \\ &= c_1\mathcal{L}[u_1] + c_2\mathcal{L}[u_2]\end{aligned}$$

Theorem (Principle of Superposition)

If u_1 and u_2 satisfy a ***linear homogeneous equation*** *($\mathcal{L}(u) = 0$), then any arbitrary linear combination, $c_1u_1 + c_2u_2$, also satisfies the* ***linear homogeneous equation****.*

Note: Concepts of linearity and homogeneity also apply to ***boundary conditions***.

Homogeneous Heat Equation

The **Heat Equation with Homogeneous Boundary Conditions**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad 0 < x < L,$$

with initial conditions (ICs) and **Dirichlet boundary conditions** (BCs):

$$u(x,0) = f(x), \quad 0 < x < L, \quad \text{with} \quad u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Separation of Variables: Developed by Daniel Bernoulli in the 1700's, we separate the temperature $u(x,t)$ into a product of a function of x and a function of t

$$u(x,t) = \phi(x)G(t)$$

Separation of Variables

Separation of Variables: With $u(x,t) = \phi(x)G(t)$, we use the ***heat equation*** and obtain:

$$\phi(x)\frac{dG}{dt} = k\frac{d^2\phi}{dx^2}G(t).$$

Separating the variables we have

$$\frac{1}{G}\frac{dG}{dt} = \frac{k}{\phi}\frac{d^2\phi}{dx^2} \qquad \text{or} \qquad \frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{d^2x}$$

Since the left hand side depends only on the independent variable t and the right hand side depends only on the independent variable x, these must equal a constant

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{d^2x} = -\lambda$$

Two ODEs

Thus, the **Separation of Variables**results in the **two ODEs**:

$$\frac{dG}{dt} = -\lambda k G \qquad \text{and} \qquad \frac{d^2\phi}{dx^2} = -\lambda\phi.$$

The **boundary conditions** with the separation assumption give:

$$u(0,t) = G(t)\phi(0) = 0 \qquad \text{or} \qquad \phi(0) = 0,$$

since we don't want $G(t) \equiv 0$. Also,

$$u(L,t) = G(t)\phi(L) = 0 \qquad \text{or} \qquad \phi(L) = 0.$$

The **Time-dependent ODE** is readily solved:

$$\begin{aligned} \frac{dG}{dt} &= -\lambda k G, \\ G(t) &= c\, e^{-k\lambda t}. \end{aligned}$$

Sturm-Liouville Problems

The **second ODE** is a **BVP** and is in a class we'll be calling **Sturm-Liouville problems**:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

Note: The **trivial solution** $\phi(x) \equiv 0$ always satisfies this **BVP**.

If we want to satisfy a nonzero initial condition, then we need to find ***nontrivial solutions*** to this BVP.

From our experience in ODEs, we can readily see there are **4 cases**:
1. $\lambda = 0$ 2. $\lambda < 0$ 3. $\lambda > 0$ 4. λ is complex

We'll ignore Case 4 and later prove that **Sturm-Liouville problems** only have **real** λ

Sturm-Liouville Problem Cases

Consider Case 1: $\lambda = 0$, so

$$\frac{d^2\phi}{dx^2} = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The general solution to this BVP is

$$\phi(x) = c_1 x + c_2.$$

We have $\phi(0) = c_2 = 0$, and $\phi(L) = c_1 L = 0 \quad \text{or} \quad c_1 = 0$.

It follows that when $\lambda = 0$, the **unique solution** to the **BVP** is the ***trivial solution***.

Sturm-Liouville Problem Cases

Consider Case 2: $\lambda = -\alpha^2 < 0$ with $\alpha > 0$, so

$$\frac{d^2\phi}{dx^2} - \alpha^2\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The general solution to this BVP is

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

We have $\phi(0) = c_1 = 0$, and $\phi(L) = c_2 \sinh(\alpha L) = 0 \quad \text{or} \quad c_2 = 0$, since $\sinh(\alpha L) > 0$.

It follows that when $\lambda < 0$, the **unique solution** to the **BVP** is the ***trivial solution***.

Sturm-Liouville Problem Cases

Consider Case 3: $\lambda = \alpha^2 > 0$ with $\alpha > 0$, so

$$\frac{d^2\phi}{dx^2} + \alpha^2\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The general solution to this BVP is

$$\phi(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x).$$

We have $\phi(0) = c_1 = 0$, and $\phi(L) = c_2\sin(\alpha L) = 0$.

It follows that either $c_2 = 0$, leading to the ***trivial solution***, or $\sin(\alpha L) = 0$.

We are interested in ***nontrivial solutions***, so we solve $\sin(\alpha L) = 0$, which occurs when $\alpha L = n\pi, n = 1, 2, ...$ or

$$\alpha = \frac{n\pi}{L}, \qquad \text{or} \qquad \lambda = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, ...$$

Eigenfunctions

We saw that if $\lambda = \alpha^2 > 0$, then the **BVP**:

$$\frac{d^2\phi}{dx^2} + \alpha^2\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0,$$

has the ***nontrivial solution***,

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, ...,$$

which are called **eigenfunctions** and their associated **eigenvalues** are given by

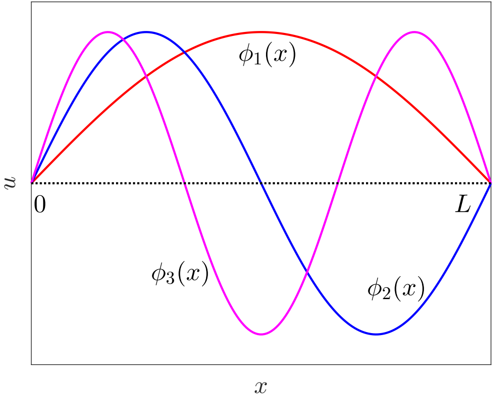
$$\lambda = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, ...$$

Note: $\phi_n(x)$ has $n - 1$ zeroes in $0 < x < L$, which later we'll prove is a general property

Eigenfunctions

The **Sturm-Liouville problem** from the **heat equation** with **Dirichlet BCs** generates a set of **eigenfunctions**, $\phi_n(x), n = 1, 2, ...$

Below is a graph of the first 3 **eigenfunctions**.



Product Solution

From above, the **Sturm-Liouville problem** from the **heat equation** gave the **eigenfunctions**:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, ...,$$

with associated **eigenvalues**

$$\lambda = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, ...$$

This can be inserted into the t-equation to give:

$$G_n(t) = B_n e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

From our separation assumption, we obtain the ***product solution***

$$u_n(x,t) = G_n(x,t)\phi_n(x) = B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, ...$$

Example

Example: Consider the **heat equation**:

PDE: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, BC: $u(0,t) = 0$,

$u(10,t) = 0.$

IC: $u(x,0) = 4\sin\left(\frac{3\pi x}{10}\right)$,

From our separation of variables results, we obtain the ***product solution***

$$u_n(x,t) = B_n e^{-\frac{kn^2\pi^2 t}{100}} \sin\left(\frac{n\pi x}{10}\right), \qquad n = 1, 2, ...$$

which satisfies the **BVP**

By inspection, we solve the **IC's** by taking $n = 3$ and $B_n = 4$. This gives the solution to this example as:

$$u(x,t) = 4e^{-\frac{9k\pi^2 t}{100}} \sin\left(\frac{3\pi x}{10}\right).$$

Example

Example 2: Vary the **IC** and consider the **heat equation**:

PDE: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$,

BC: $u(0,t) = 0$,
$u(5,t) = 0$.

IC: $u(x,0) = 3\sin\left(\frac{3\pi x}{5}\right) + 7\sin(\pi x)$,

With the **Principle of Superposition**, we can add our ***product solutions***, $u_3(x,t) + u_5(x,t)$.

By inspection, we satisfy the **IC's** by taking $B_3 = 3$ and $B_5 = 7$. This gives the solution to this example as:

$$u(x,t) = 3e^{-\frac{9k\pi^2 t}{25}} \sin\left(\frac{3\pi x}{5}\right) + 7e^{-k\pi^2 t} \sin(\pi x).$$

Extended Superposition Principle

Extended Superposition Principle: The superposition principle can be extended to show that if u_1, u_2, ..., u_M, are solutions of a linear homogeneous problem, then any ***linear combination***

$$c_1 u_1 + c_2 u_2 + \ldots + c_M u_M,$$

is also a solution.

It follows for the ***homogeneous heat problem***

$$u_t = k u_{xx}, \qquad u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0,$$

that we can write a solution of the form

$$u(x,t) = \sum_{n=1}^{M} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

Heat Problem with ICs

The complete ***homogeneous heat problem*** includes an **IC**.

Again the solution has the form:

$$u(x,t) = \sum_{n=1}^{M} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right),$$

and will satisfy any **IC**, where

$$u(x,0) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x),$$

i.e., any **IC** that is a finite sum of sine functions.

What can we do about solving an arbitrary $f(x)$?

Arbitrary ICs

What if $f(x)$ is NOT a finite linear combination of appropriate sine functions?

Soon we'll learn about Fourier series

1. Any function with reasonable restrictions can be approximated by a linear combination of $\sin\left(\frac{n\pi x}{L}\right)$
2. The approximation improves with M increasing
3. If we consider the limit as $M \to \infty$, then with some restrictions the eigenfunctions, $\sin\left(\frac{n\pi x}{L}\right)$ in the right combination converges to $f(x)$
4. It remains to find the constants, B_n, such that:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Orthogonality of Sines

Assume $m \neq n$, integers and with some trig identities consider

$$\begin{aligned}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx &= \int_0^L \frac{\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)}{2}dx \\ &= \frac{1}{2}\left(\frac{\sin\left(\frac{(n-m)\pi x}{L}\right)}{(n-m)\pi/L} - \frac{\sin\left(\frac{(n+m)\pi x}{L}\right)}{(n+m)\pi/L}\right)\Bigg|_0^L \\ &= 0\end{aligned}$$

When $m = n$, then

$$\begin{aligned}\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx &= \int_0^L \frac{1 - \cos\left(\frac{2n\pi x}{L}\right)}{2}dx \\ &= \left(\frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{L}\right)}{4n\pi/L}\right)\Bigg|_0^L \\ &= \frac{L}{2}\end{aligned}$$

Orthogonality

Definition (Orthogonality - Function Inner Product)

Whenever

$$\int_0^L A(x)B(x)dx = 0,$$

we say that the functions, $A(x)$ and $B(x)$ are **orthogonal** over the interval $[0, L]$.

Previous slide shows that the set of functions, $\sin\left(\frac{n\pi x}{L}\right), n = 1, 2, ...,$ are **orthogonal** to each other

This ***orthogonal set of functions*** arise from the **eigenvalue BVP**:

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

Later generalize this property to any **Sturm-Liouville Problem**

Finding B_n

Consider the expression

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Use the ***orthogonality of these sine functions***, so multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate $x \in [0, L]$

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \left(\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

To use orthogonality requires some analysis to allow the interchange of the integration and summation

Finding B_n

Assuming we can interchange the integration and summation,

$$\begin{aligned}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx &= \sum_{n=1}^{\infty} B_n \int_0^L \left(\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\right)dx \\ &= B_m\left(\frac{L}{2}\right),\end{aligned}$$

by the orthogonality of the sine functions

If follows that we can obtain the appropriate coefficients (Fourier) to represent an arbitrary function $f(x)$,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Heat Equation Example

Example: Consider the equation:

PDE: $u_t = ku_{xx}, \qquad t > 0, \qquad 0 < x < L,$

BC: $u(0,t) = 0, \qquad u(L,t) = 0, \qquad t > 0,$

IC: $u(x,0) = 100, \qquad 0 < x < L$

From before, the solution satisfies:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

The **Fourier coefficients** are given by

$$B_n = \frac{2}{L}\int_0^L 100 \sin\left(\frac{n\pi x}{L}\right) dx.$$

Heat Equation Example

Expanding the **Fourier coefficients**:

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L 100\sin\left(\frac{n\pi x}{L}\right)dx = \frac{200}{L}\left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{200}{n\pi}\left(1-\cos(n\pi)\right) = \begin{cases} \frac{400}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \end{aligned}$$

Thus, the solution satisfies:

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n}e^{-\frac{kn^2\pi^2 t}{L^2}}\sin\left(\frac{n\pi x}{L}\right).$$

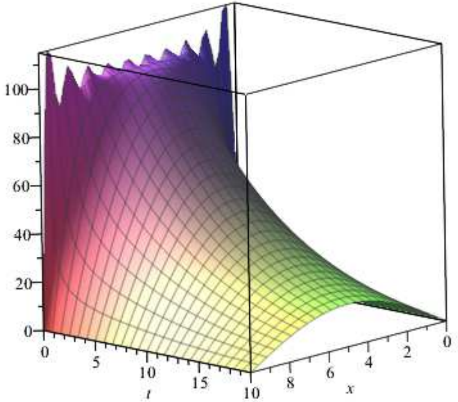
Because of the coefficient on the exponential decay term, this solution rapidly approaches

$$u(x,t) \approx \frac{400}{\pi}e^{-\frac{k\pi^2 t}{L^2}}\sin\left(\frac{\pi x}{L}\right).$$

Heat Equation with Maple

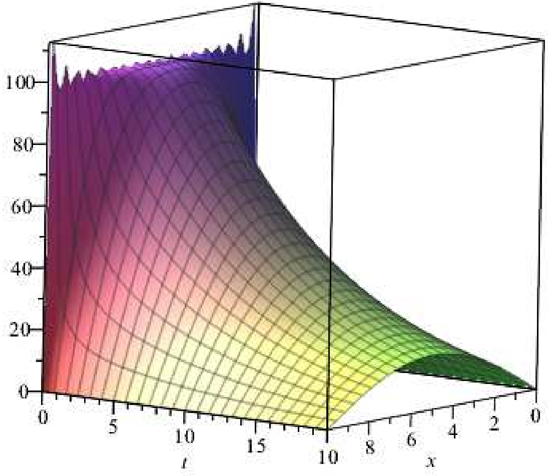
Heat Equation with Maple: Show commands and plots.

```
> u := (x,t) -> (200/Pi)*sum(((1-(-1)^n)/n)*sin(n*Pi*x/10)
            *exp(-(n*Pi/10)^2*t),n=1..20);
> plot3d(u(x,t),x=0..10,t=0..20);
```



Heat Equation with Maple

Heat Equation with Maple: Increase the sum to 60 (30 nonzero terms)



Heat Equation with MatLab

MatLab Program for Heat equation solution $u(x,t)$

```
% Solutions to the heat flow equation
% on one-dimensional rod length L
format compact;
L = 10;                 % length of rod
Temp = 100;             % Constant temperature of ...
    rod, initially
tfin = 20;              % final time
k = 1;              % heat coef of the medium
x=linspace(0,L,151);
t=linspace(0,tfin,151);
[X,T]=meshgrid(x,t);

b=zeros(1,200);
U=zeros(NptsT,NptsX);
```

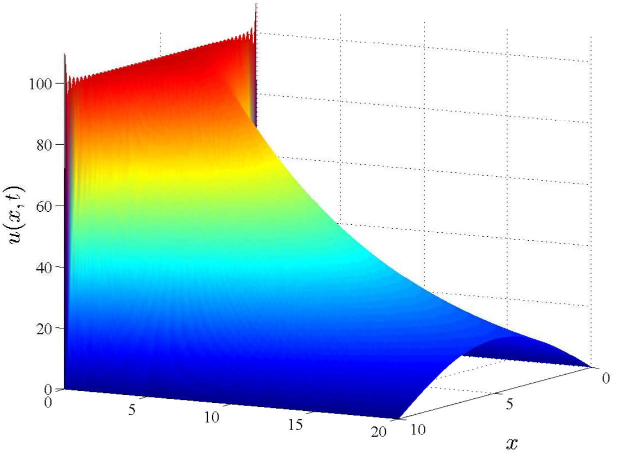
Heat Equation with MatLab

```
for n=1:200
    b(n)=(2*Temp/(n*pi))*(1-(-1)^n);   % Fourier coefficients
    Un=b(n)*exp(-(n*pi*k/L)^2*T).*sin(n*pi*X/L);   % Temperature(n)
    U=U+Un;
end

set(gca,'FontSize',[14]);
surf(X,T,U);
shading interp
xlabel('$x$','Fontsize',14,'interpreter','latex');
ylabel('$t$','Fontsize',14,'interpreter','latex');
zlabel('$u(x,t)$','Fontsize',14,'interpreter','latex');
axis tight;
view([120 10]);
print -depsc heat_surf.eps
```

Heat Equation with MatLab

Graph of Heat Equation Solution using 200 terms with MatLab



Heat Equation with MatLab

Changing the view to `view([0 90]);`, obtain a ***heat map***