Section 1.2: Row Reduction and Echelon Forms

We defined Echelon (ref) and Reduced Echelon (rref) forms for a matrix. The rref for a matrix is unique. We then defined

- a **pivot position** is an entry in a matrix where a leading 1 would be in the rref,
- a **pivot column** is a column that contains a pivot position.

Row Reduction Algorithm

To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$$

We did the following steps

Step 1: The left most column is a pivot column. The top position is a pivot position.

Step 2: Get a nonzero entry in the top left position by row swapping if needed. ($R_1 \leftrightarrow R_3$)

Step 3: Use row operations to get zeros in all entries below the pivot.

$-R_1 + R_3 \to R_3$

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3.

We then scaled R_2 ($\frac{1}{2}R_2 \to R_2$), and then cleared out the entries in the second pivot's column below the 1 ($-3R_2 + R_3 \to R_3$) to get

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is an echolon form we ended up with.

Row Reduction Algorithm

To obtain a reduced row echelon form:

Step 5: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

we don't have to scale row 3 because the pivot position has a 1 in it.

Clear out the entries above this 1.

$-R_3 + R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$-6R_3 + R_1 \to R_1$

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 0 & -9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

we already have a 1 in the 2nd row, 2nd column

Clear out the entries above this.

$9R_2 + R_1 \to R_1$

$$\begin{bmatrix} 3 & 0 & -6 & 0 & 9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Move to the left. We need the next pivot to be a 1.

$\frac{1}{3} R_1 \to R_1$

Row Reduction Algorithm

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is the rref for the matrix.

Echelon Form & Solving a System

Remark: The row operations used to get an rref correspond to an **equivalent** system!

Consider the reduced echelon matrix, and describe the solution set for the associated system of equations (the one who'd have this as its augmented matrix).

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system would have 4 equations in 5 variables.

$x_1 + x_2 = 3$

$x_3 - 2x_5 = 4$

$x_4 = -9$

$x_1 = 3 - x_2$

x_2 is anything

$x_3 = 4 + 2x_5$

$x_4 = -9$

x_5 is anything

we'll call x_2 and x_5 free variables

(these had non pivot columns)

we'll call x_1, x_3, and x_4 basic variables

(those had pivot columns)

Consistent versus Inconsistent Systems

Consider each rref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Consistent

$x_1 + 2x_2 = 0$

$x_3 = 4$

$x_1 = -2x_2$

x_2 - free

$x_3 = 4$

∞-many solutions

Consistent

$x_1 = 0$

$x_2 = 4$

$x_3 = -3$

exactly one solution

Inconsistent

$x_1 + 2x_3 = 3$

$x_2 + x_3 = 0$

$0 = 1$

This last equation is always false.

An Existence and Uniqueness Theorem

Theorem: A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0\ 0\ \cdots\ 0\ b], \quad \text{for some nonzero } b.$$

If a linear system is consistent, then it has

(i) exactly one solution if there are no free variables (no row of all zeros), or

(ii) infinitely many solutions if there is at least one free variable (at least one row of all zeros).