Math 5510 - Partial Differential Equations

Fourier Series

Ahmed Kaffel,
⟨ahmed.kaffel@marquette.edu⟩

Department of Mathematical and Statistical Sciences
Marquette University

https://www.mscsnet.mu.edu/~ahmed/teaching.html

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Outline

1. Introduction
   - Definitions
   - Convergence Theorem
   - Example
2. Fourier Sine and Cosine Series
   - Gibbs Phenomenon
   - Continuous Fourier Series
3. Differentiation of Fourier Series
   - Differentiation of Fourier Series
   - Differentiation of Cosine Series
   - Differentiation of Sine Series
4. Method of Eigenfunction Expansion

Introduction

The **separation of variables** technique solved our various **PDEs** provided we could write:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

Questions:

1. Does the infinite series converge?
2. Does it converge to $f(x)$?
3. Is the resulting infinite series really a solution of the PDE (and its subsidiary conditions)?

Mathematically, these are all difficult problems, yet these solutions have worked well since the early 1800's.

Definitions

Begin by restricting the class of $f(x)$ that we'll consider.

Definition (Piecewise Smooth)

A function $f(x)$ is ***piecewise smooth*** on some interval if and only if $f(x)$ is continuous and $f'(x)$ is continuous on a finite collection of sections of the given interval.

The only discontinuities allowed are jump discontinuities.

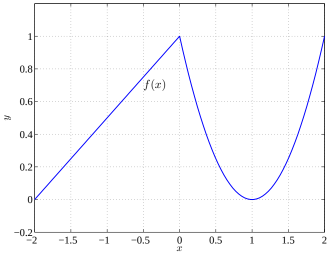
Definition (Jump Discontinuity)

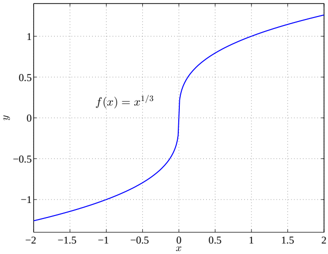
A function $f(x)$ has a ***jump discontinuity*** at a point $x = x_0$, if the limit from the right $[f(x_0^+)]$ and the limit from the left $[f(x_0^-)]$ both exist and are not equal.

Piecewise smooth allows only a finite number of ***jump discontinuities*** in the function, $f(x)$, and its derivative, $f'(x)$.

Piecewise Smooth

The graph on the left is **piecewise smooth** with the function being continuous, but having a ***jump discontinuity*** in the derivative at $x = 0$



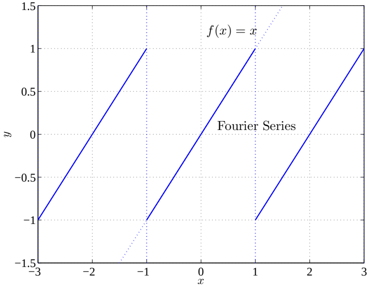


The graph on the right is **not piecewise smooth**, as the derivative becomes unbounded in any neighborhood of $x = 0$

Periodic Extension

The **Fourier series of $f(x)$** on an interval $-L \leq x \leq L$ is periodic with **period $2L$**.

However, the function $f(x)$ itself doesn't need to be periodic.



The graph above gives the **Fourier series period 2 extension** of $f(x) = x$ (along with $f(x)$, not periodic).

Fourier Series

Definitions of Fourier coefficients and a Fourier series. We must distinguish between a function $f(x)$ and its Fourier series over the interval $-L \leq x \leq L$.

$$\text{Fourier series} = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

The infinite series may not converge, and if it converges, it may not converge to $f(x)$

If the series converges, the **Fourier coefficients** a_0, a_n, and b_n use certain **orthogonality integrals**.

Fourier coefficients

Definition (Fourier coefficients)

The definition of the **Fourier coefficients** are:

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}$$

The coefficients must be defined, *e.g.*, $\left|\int_{-L}^{L} f(x)dx\right| < \infty$ for a_0 to exist. (No Fourier series for $f(x) = 1/x^2$.)

Fourier convergence

We write the **Fourier series**

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Theorem (Fourier convergence)

If $f(x)$ is ***piecewise smooth*** *on the interval $-L \leq x \leq L$, then the* ***Fourier series*** *of $f(x)$ converges to:*

1. *The periodic extension of $f(x)$, where the periodic extension is continuous*
2. *The average of the two limits, usually $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$, where the periodic extension has a* ***jump discontinuity***

Proof: The **proof of this theorem** requires significant techniques from Mathematical analysis, which is beyond the scope of this course.

Example

Example: Consider the Heaviside function shifted by 1:

$$f(x) = H(x-1) = \begin{cases} 0, & x < 1, \\ 1, & x \geq 1. \end{cases}$$

Find the Fourier series with $L = 2$.

The Fourier constant coefficient is

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{1}^{2} 1\ dx = \frac{1}{4}.$$

The cosine coefficients:

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{1}^{2}\cos\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{\sin(n\pi) - \sin(n\pi/2)}{n\pi} = -\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

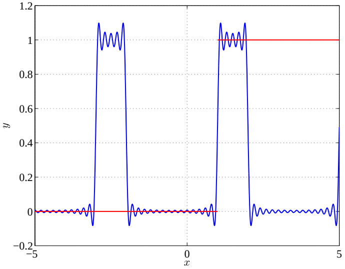
Example 2

The sine coefficients:

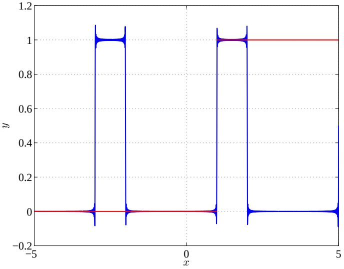
$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{1}^{2}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{\cos(n\pi/2) - \cos(n\pi)}{n\pi} = \frac{1}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right).$$

The function, $f(x)$, and truncated Fourier series.



Fourier series, $n = 20$



Fourier series, $n = 200$

Example

3

```
% Periodic Fourier series, -2 < x < 2
% Step function at x = 1

NptsX=2000;              % number of x pts
Nf=200;                  % number of Fourier terms
x=linspace(-5,5,NptsX);

a0=1/4;
a=zeros(1,Nf);
b=zeros(1,Nf);
f=a0*ones(1,NptsX);

for n=1:Nf
    a(n)= -sin(n*pi/2)/(n*pi); % Fourier cosine ...
        coefficients
    b(n)=(cos(n*pi/2)-cos(n*pi))/(n*pi);  % ...
        Fourier sine coefficients
```

Example


```
    fn=a(n)*cos((n*pi*x)/2) + ...
        b(n)*sin((n*pi*x)/2);  % Fourier function(n)
    f=f+fn;
end
set(gca,'FontSize',16);
plot(x,f,'b-','LineWidth',1.5);
hold on
plot([-5,1],[0,0],'r-','LineWidth',1.5);
plot([1,5],[1,1],'r-','LineWidth',1.5);
xlabel('$x$','FontSize',16,'FontName',fontlabs, ...
    'interpreter','latex');
ylabel('$y$','FontSize',16,'FontName',fontlabs, ...
    'interpreter','latex');
axis on; grid;

print -depsc eg200_gr.eps
```

Fourier Sine Series

If $f(x)$ is an **odd function**, then $a_0 = a_n = 0$ and only the sine series remains:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

This series appeared for solutions of the ***heat equation***, $0 < x < L$ with $u(0,t) = u(L,t) = 0$

The **Sine series** produces an **odd extension** of $f(x)$

$$\begin{aligned} f(x) &\sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L, \\ B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Fourier Cosine Series

If $f(x)$ is an **even function**, then $b_n = 0$ and only the cosine series remains:

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 < x < L,$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \qquad \text{and} \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

This series appeared for solutions of the ***heat equation***, $0 < x < L$ with $u_x(0,t) = u_x(L,t) = 0$.

Gibbs Phenomenon 1

Let $f(x) = 100$, and consider the **odd extension** of this function, so $f(x)$ is defined by

$$f(x) = \begin{cases} 100, & 0 < x < L, \\ -100, & -L < x < 0. \end{cases}$$

and extend it periodically with period $2L$.

As an **odd function**, this has a **Fourier sine series**

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

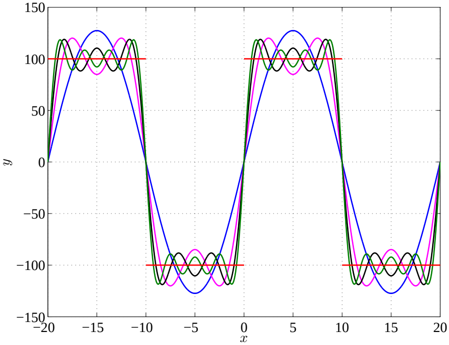
with

$$B_n = \frac{2}{L}\int_0^L 100 \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} \frac{400}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Gibbs Phenomenon

We examine the graph for $n = 1, 3, 5, 7$ of

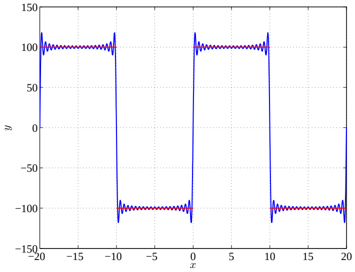
$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{with} \quad B_n = \begin{cases} \frac{400}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$



Gibbs Phenomenon 3

We examine the graphs for $n = 40$ (20 nonzero terms) and $n = 200$ (100 nonzero terms) for

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{with} \quad B_n = \begin{cases} \frac{400}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$



$n = 40$ $n = 200$

Gibbs Phenomenon

The **Fourier series** for the $2L$-periodic, **odd extension** of $f(x) = 100$,

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{with} \quad B_n = \begin{cases} \frac{400}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

It is clear that the **Fourier series** converges to **0** at $x = 0$ as every term in the series is **0**.

Similarly, the **Fourier series** converges to **0** at any $x = nL$ for $n = 0, \pm1, \pm2, ...$, as every term in the series is also **0**.

The **Fourier Convergence Theorem** claims that the series converges to **100** for each $0 < x < L$.

Gibbs Phenomenon

The $2L$-periodic, **odd extension** of $f(x) = 100$,

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{with} \quad B_n = \begin{cases} \frac{400}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

by the **Fourier Convergence Theorem** converges to **100** for $0 < x < L$, which is hard to show for most values of x.

Consider $x = \frac{L}{2}$,

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{2}\right) = \frac{400}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right)$$

Euler's formula gives $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$, (which is a very inefficient way to compute π, as it is an alternating series that does not ***converge absolutely***)

Gibbs Phenomenon

Harder to show convergence for other values of $x \in (0, L)$.

Convergence easily visualized as worst near **jump discontinuity**

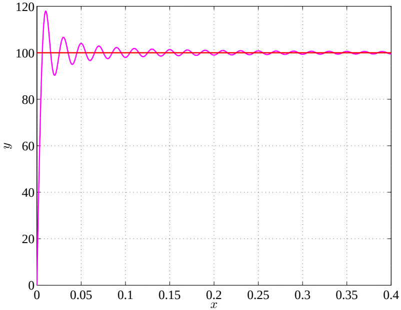
For any finite sum in the series near $x = 0$, the solution starts at **0**, then shoots up beyond 100, the primary overshoot

Examine previous $f(x)$

Figure (close up) with $n = 1000$ (or 500 nonzero terms)

The overshoot is about 20%

The maximum occurs at $(0.01, 117.898)$



Gibbs Phenomenon 7

This **overshoot** is an example of the **Gibbs phenomenon**

For large n, in general, there is an overshoot of approximately 9% of the jump discontinuity

Note the previous example had a jump of **200**, and we saw the maximum of **117.898**, which is 9% of the jump

The **Gibbs phenomenon** only occurs for a finite series at a **jump discontinuity**

Continuous Fourier Series

Theorem (Fourier Series)

For a piecewise smooth $f(x)$, the ***Fourier series*** *of $f(x)$ is continuous and converges to $f(x)$ for $x \in [-L, L]$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.*

Theorem (Fourier Cosine Series)

For a piecewise smooth $f(x)$, the ***Fourier cosine series*** *of $f(x)$ is continuous and converges to $f(x)$ for $x \in [0, L]$ if and only if $f(x)$ is continuous.*

Theorem (Fourier Sine Series)

For a piecewise smooth $f(x)$, the ***Fourier sine series*** *of $f(x)$ is continuous and converges to $f(x)$ for $x \in [0, L]$ if and only if $f(x)$ is continuous and both $f(0) = 0$ and $f(L) = 0$.*

Differentiation of Fourier Series

Previously, we solved

PDE: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, BC: $u(0,t) = 0$, $u(L,t) = 0$.

IC: $u(x,0) = f(x)$,

and obtained the solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

The **Superposition principle** justified this solution for any ***finite series***, but can it be extended to the ***infinite series***?

If $f(x)$ is piecewise smooth, then the **Fourier Convergence Theorem** shows that the **Fourier series** converges to the **Initial Conditions**

Differentiation of Fourier Series

Suppose we can differentiate the series term-by-term, then in t

$$\frac{\partial u}{\partial t} = -\sum_{n=1}^{\infty} \frac{kn^2\pi^2}{L^2} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

Taking two partials with respect to x gives

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

It follows that our solution above satisfies the **heat equation**:

$$u_t = ku_{xx}.$$

Counterexample 1

Differentiation Counterexample: Consider the **Fourier sine series** for $f(x) = x$ with $x \in [0, L]$:

$$x \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

The **Fourier coefficients** satisfy:

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{2L}{n^2\pi^2}\left(\sin\left(\frac{n\pi\, x}{L}\right) - \frac{n\pi\, x}{L}\cos\left(\frac{n\pi\, x}{L}\right)\right)\Bigg|_0^L \\ &= -\frac{2L}{n\pi}\cos(n\pi) = \frac{2L}{n\pi}(-1)^{n+1} \end{aligned}$$

Thus, we have

$$x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right), \qquad x \in [0, L).$$

Counterexample 2

Differentiation Counterexample: Continuing with

$$x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right), \qquad x \in [0, L),$$

we differentiate the series term-by-term and obtain:

$$2\sum_{n=1}^{\infty}(-1)^{n+1} \cos\left(\frac{n\pi x}{L}\right).$$

However, the series above is clearly not the cosine series for $f'(x) = 1$ (the derivative of x)

This series fails to converge anywhere, since the n^{th} term doesn't approach zero!

Differentiation of Fourier Series

When is term-by-term differentiation justified?

Theorem (Term-by-Term Differentiation)

*A **Fourier series** that is continuous can be differentiated term-by-term if $f'(x)$ is **piecewise smooth**.*

Corollary

*If $f(x)$ is **piecewise smooth**, then the **Fourier series** of a continuous function, $f(x)$ can be differentiated term-by-term if $f(-L) = f(L)$.*

Differentiation of Fourier Cosine Series

From our earlier result, if $f(x)$ is continuous, then its Fourier cosine series is continuous, avoiding ***jump discontinuities*** where difficulties occur for term-by-term differentiation

Theorem (Cosine Series Term-by-Term Differentiation)

*If $f'(x)$ is **piecewise smooth**, then a continuous **Fourier cosine series** of $f(x)$ can be differentiated term-by-term.*

Corollary (Cosine Series Term-by-Term Differentiation)

*If $f'(x)$ is **piecewise smooth**, then the **Fourier cosine series** of a continuous function $f(x)$ can be differentiated term-by-term.*

Cosine Series Term-by-Term Differentiation

Thus, if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \qquad 0 \leq x \leq L,$$

where equality implies convergence for all $0 \leq x \leq L$, the theorem above implies that

$$f'(x) \sim -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) A_n \sin\left(\frac{n\pi x}{L}\right).$$

This sine series converges to points of continuity of $f'(x)$ and to the average where the Fourier sine series of $f'(x)$ is discontinuous.

Cosine Example 1

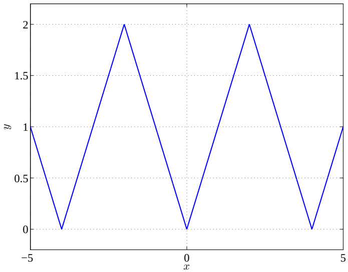
Example: Consider $f(x) = x$ on $0 \le x \le L$. Create an even extension, then make this $2L$-periodic as seen in the graph.

The function has a continuous, piecewise smooth Fourier cosine series.

By our theorem, this **Fourier series** converges

The **Fourier coefficients** are

$$A_0 = \frac{1}{L}\int_0^L x dx = \frac{x^2}{2L}\bigg|_0^L = \frac{L}{2}$$



and

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \left(\frac{2L}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right) + \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\bigg|_0^L \\ &= \frac{2L}{n^2\pi^2}\left((-1)^n - 1\right) \end{aligned}$$

Cosine Example 2

Thus,

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos\left(\frac{n\pi x}{L}\right),$$

where the series converges pointwise to the graph on the previous slide.

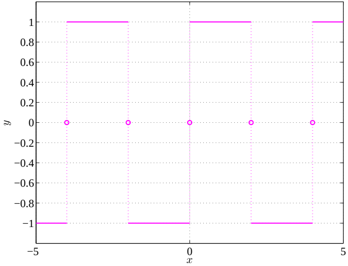
Note: This series converges absolutely by comparison to the series for $\frac{1}{n^2}$

The derivative of $f(x)$ is piecewise constant, as seen in the graph (right).

Differentiating term-by-term gives

$$1 \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right), \qquad 0 < x < L.$$

The weaker series convergence is easily seen, and it is easy to verify that this is the sine series for $f'(x) = 1$.



Sine Series Term-by-Term Differentiation

Similar results hold for the **sine series** with more conditions

Theorem

Sine Series Term-by-Term Differentiation] If $f'(x)$ is ***piecewise smooth****, then a continuous* ***Fourier sine series*** *of $f(x)$ can be differentiated term-by-term.*

Corollary (Sine Series Term-by-Term Differentiation)

If $f'(x)$ is ***piecewise smooth****, then the* ***Fourier sine series*** *of a continuous function $f(x)$ can be differentiated term-by-term if $f(0) = 0$ and $f(L) = 0$.*

Sine Series Term-by-Term Differentiation

Proof: We prove term-by-term differentiation of the ***Fourier sine series*** of a ***continuous*** function $f(x)$, when $f'(x)$ is ***piecewise smooth*** and $f(0) = 0 = f(L)$:

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

where B_n are expressed later. Equality holds if $f(0) = 0 = f(L)$.

If $f'(x)$ is piecewise smooth, then $f'(x)$ has a ***Fourier cosine series***

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

where A_0 and A_n are expressed later.

This series will not converge to $f'(x)$ at points of ***discontinuity***.

Sine Series Term-by-Term Differentiation

Proof (cont): Need to verify that

$$f'(x) \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n \cos\left(\frac{n\pi x}{L}\right).$$

The **Fundamental Theorem of Calculus** gives:

$$A_0 = \frac{1}{L}\int_0^L f'(x)dx = \frac{1}{L}\Big(f(L) - f(0)\Big).$$

Integrating by parts,

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f'(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left[f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right] \end{aligned}$$

Sine Series Term-by-Term Differentiation

Proof (cont): However, B_n, the ***Fourier sine series coefficient*** of $f(x)$ is

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx,$$

so for $n \neq 0$

$$A_n = \frac{n\pi}{L}B_n + \frac{2}{L}\Big[(-1)^n f(L) - f(0)\Big].$$

It follows that we need $f(0) = 0 = f(L)$ for both $A_0 = 0$ and $A_n = \frac{n\pi}{L}B_n$, completing the proof.

However, this proof gives us more information about ***differentiating the Fourier sine series***.

Sine Series Term-by-Term Differentiation

The more general theorem for ***differentiating the Fourier sine series*** is below:

Theorem

If $f'(x)$ is ***piecewise smooth****, then the* ***Fourier sine series*** *of a continuous function $f(x)$,*

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

cannot, in general be differentiated term-by-term. However,

$$f'(x) \sim \frac{1}{L}\Big[f(L) - f(0)\Big] + \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}B_n + \frac{2}{L}\Big[(-1)^n f(L) - f(0)\Big]\right)\cos\left(\frac{n\pi x}{L}\right).$$

Sine Series Term-by-Term Differentiation

Example: Previously considered $f(x) = x$ with a ***Fourier sine series*** and showed this could not be differentiated term-by-term. The **Fourier sine series** satisfies:

$$f(x) = x \sim 2\sum_{n=1}^{\infty} \frac{L(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{L}\right).$$

Since $f(0) = 0$ and $f(L) = L$, from the general formula above:

$$A_0 = \frac{1}{L}\Big(f(L) - f(0)\Big) = 1.$$

and

$$\begin{aligned} A_n &= \frac{n\pi}{L} B_n + \frac{2}{L}\Big[(-1)^n f(L) - f(0)\Big] \\ &= 2(-1)^{n+1} + 2(-1)^n \ = \ 0. \end{aligned}$$

It follows that we obtain the correct derivative

$$f'(x) = 1.$$

Method of Eigenfunction Expansion

Want to apply techniques of ***differentiating a Fourier series*** term-by-term to **PDEs**

Use an alternative **method of eigenfunction expansion**, which can be applied to **nonhomogeneous BCs**

Consider an ***eigenfunction expansion*** of the form

$$u(x,t) \sim \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

where the ***Fourier sine coefficients*** depend on time, t

Method of Eigenfunction Expansion

The initial condition, $u(x,0) = f(x)$, is satisfied if

$$f(x) \sim \sum_{n=1}^{\infty} B_n(0) \sin\left(\frac{n\pi x}{L}\right),$$

where the initial ***Fourier sine coefficients*** are

$$B_n(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Can we differentiate term-by-term to satisfy the **heat equation**,

$$u_t = k u_{xx}?$$

Need **two** partial derivatives with respect to x and **one** partial derivative with respect to t.

Method of Eigenfunction Expansion

If $u(x,t)$ is continuous, then the ***Fourier sine series*** can be differentiated term-by-term provided

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

(homogeneous BCs)

The result is

$$\frac{\partial u}{\partial x} \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos\left(\frac{n\pi x}{L}\right),$$

which is a ***Fourier cosine series***

Provided $\frac{\partial u}{\partial x}$ is continuous, it can be differentiated term-by-term:

$$\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} B_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

Method of Eigenfunction Expansion

The **two** derivatives w.r.t. x could be taken term-by-term provided the problem has homogeneous BCs.

Need

$$\frac{\partial u}{\partial t} \sim \sum_{n=1}^{\infty} \frac{dB_n}{dt} \sin\left(\frac{n\pi x}{L}\right).$$

If term-by-term evaluation is justified, then

$$\frac{dB_n}{dt} = -k\frac{n^2\pi^2}{L^2}B_n(t),$$

so

$$B_n(t) = B_n(0)e^{-\frac{n^2\pi^2}{L^2}kt}.$$

Method of Eigenfunction Expansion

Theorem

*The **Fourier series** of a continuous function $u(x,t)$*

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} \left(a_n(t) \cos\left(\frac{n\pi x}{L}\right) + b_n(t) \sin\left(\frac{n\pi x}{L}\right)\right),$$

can be differentiated term-by-term with respect to t

$$\frac{\partial u(x,t)}{\partial t} = a_0'(t) + \sum_{n=1}^{\infty} \left(a_n'(t) \cos\left(\frac{n\pi x}{L}\right) + b_n'(t) \sin\left(\frac{n\pi x}{L}\right)\right),$$

*if $\frac{\partial u}{\partial t}$ is **piecewise smooth**.*

This theorem justifies the use of separation of variables and our solution.

Term-by-Term Integration

Theorem

*A **Fourier series** of a piecewise smooth $f(x)$ can always be integrated term-by-term and the result is a convergent infinite series that always converges to the integral of $f(x)$ for $-L \leq x \leq L$ (even if the original Fourier series has **jump discontinuities**.*