Section 1.3: Vector Equations

We defined

- vectors in $\mathbb{R}^n$,
- the operations of scalar multiplication and vector addition in $\mathbb{R}^n$,
- vector equivalence,
- a linear combination of vectors, and
- Span.

We introduced notation:

Letting $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$ be n vectors in $\mathbb{R}^m$, we can write an $m \times n$ matrix having these vectors as columns

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Span: Three Equivalent Things

1. If $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$.

2. From the previous result, we know this is equivalent to saying that the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

   has a solution.

3. This is in turn the same thing as saying the linear system with augmented matrix $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$ is consistent.

Examples

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

We found that $\mathbf{b}$ is not in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$. We did this by setting up the matrix and obtaining an rref:

$$[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}] \quad \text{(rref)} \longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The last column is a pivot column. The system is inconsistent.

(b) Determine if $\mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

Do there exist scalars c_1, c_2 such that

$c_1 \vec{a}_1 + c_2 \vec{a}_2 = \vec{b}$?

We can answer by considering the matrix $[\vec{a}_1\ \vec{a}_2\ \vec{b}]$.

$$[\vec{a}_1\ \vec{a}_2\ \vec{b}] = \begin{bmatrix} 1 & -1 & 5 \\ 1 & 4 & -5 \\ 2 & -2 & 10 \end{bmatrix} \xrightarrow[\text{(TI-89)}]{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The 3rd column is not a pivot column. Hence, the system is consistent. $\vec{b}$ is in $\text{Span}\{\vec{a}_1, \vec{a}_2\}$.

From the rref, the system is equivalent to

$$\left.\begin{aligned} 1c_1 + 0c_2 &= 3 \\ 0c_1 + 1c_2 &= -2 \\ 0c_1 + 0c_2 &= 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} c_1 &= 3 \\ c_2 &= -2 \end{aligned}$$

Note $3\vec{a}_1 + (-2)\vec{a}_2 = 3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$

Another Example

Give a geometric description of the subset of $\mathbb{R}^2$ given by

$\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$. If $\vec{u}$ is in span $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ then

$\vec{u} = c\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}c\\0\end{bmatrix}$ for some real number c.

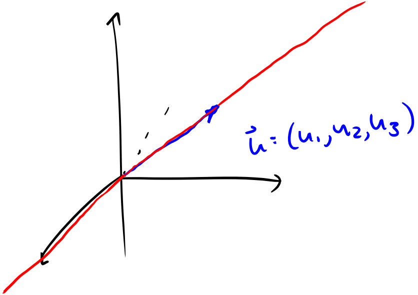
Like a traditional point this is $(c, 0)$ for some real number c.

This is the x-axis.

Span$\{\mathbf{u}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ is any nonzero vector in $\mathbb{R}^3$, then Span$\{\mathbf{u}\}$ is a line through the origin parallel to $\mathbf{u}$.

These are the vectors $c\vec{u}$ in $\mathbb{R}^3$ for all possible numbers c.



Span$\{\vec{u}\}$ is a line through the origin and the point (u_1, u_2, u_3).

Span$\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero, and nonparallel vectors in $\mathbb{R}^3$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a plane containing the origin parallel to both vectors.

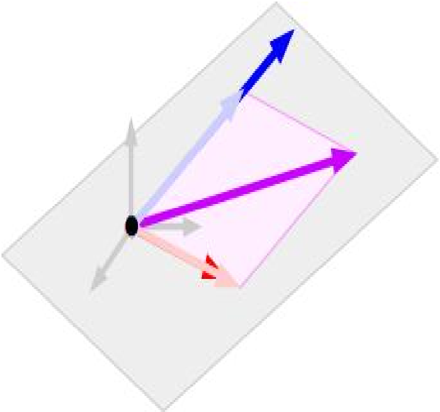


Figure: The red and blue vectors are $\mathbf{u}$ and $\mathbf{v}$. The plane is the collection of all possible linear combinations. (A purple representative is shown.)

Example

Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (0, 2)$ in $\mathbb{R}^2$. Show that for every pair of real numbers a and b, that (a, b) is in Span$\{\mathbf{u}, \mathbf{v}\}$.

We need to show that for any (a,b), there exist weights x_1 and x_2 such that

$$x_1\vec{u} + x_2\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

We can use an augmented matrix $\left[\vec{u} \;\; \vec{v} \;\; \begin{bmatrix} a \\ b \end{bmatrix}\right]$

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 2 & b \end{bmatrix}$$

Do row reduction to rref.

$-R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 2 & b-a \end{bmatrix} \quad \tfrac{1}{2}R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$$ The last column is not a pivot column for any (a,b).

The system is consistant for all (a,b).

That is, (a,b) is in $\text{Span}\{\vec{u}, \vec{v}\}$.

Moreover,

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b-a}{2} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$