Math 5510 - Partial Differential Equations

Vibrating String

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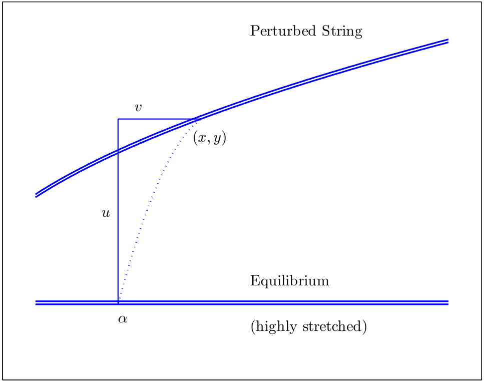
Outline

Introduction

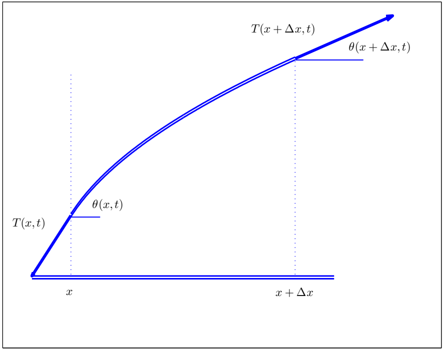
An important application of **PDEs** is the investigation of **vibrations** of perfectly elastic strings and membranes



- Consider a particle at position α in a highly stretched string
- Assume a small displacement as seen above

Derivation

Simplify by assuming the displacement is only vertical, $y = u(x,t)$



- Apply **Newton's Law** to an infinitesimally small segment of string between x and $x + \Delta x$
- Assume string has ***mass density*** $\rho_0(x)$, so ***mass*** is $\rho_0(x)\Delta x$

Derivation

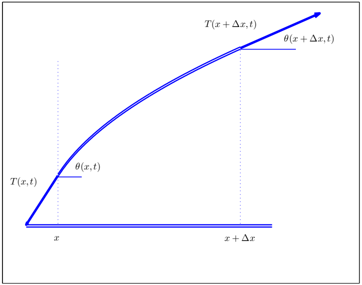
Newton's Law acting on string considers all forces

Forces include gravity, resistance, and tension - "body" forces

Assume string is ***perfectly flexible***, so no bending resistance

This implies primary force is tangent to the string at all points

Tension is the tangential force with



$$\frac{dy}{dx} = \frac{\partial u}{\partial x} = \tan(\theta(x,t))$$

Derivation

Newton's Law gives $\tilde{\mathbf{F}} = m\tilde{\mathbf{a}}$, which is

$$\begin{aligned} \rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2} &= T(x+\Delta x, t)\sin(\theta(x+\Delta x, t)) \\ &\quad -T(x,t)\sin(\theta(x,t)) + \rho_0(x)\Delta x Q(\xi, t), \end{aligned}$$

where $\xi \in [x, x+\Delta x]$ and $Q(\xi, t)$ are any "body" accelerations, such as gravity or air resistance.

Dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\sin(\theta(x,t))\Big) + \rho_0(x)Q(x,t).$$

For θ "small," let

$$\frac{\partial u}{\partial x} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \approx \sin(\theta)$$

String Equation

From previous results, obtain **String Equation**

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t)\frac{\partial u}{\partial x}\right) + \rho_0(x)Q(x,t).$$

If the string is perfectly elastic, then $T(x,t) \approx T_0$ constant, which is equivalent to almost uniform stretching along string

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \rho_0(x)Q(x,t).$$

If the **body force** is small and **density** is constant, then

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$

where $c^2 = \frac{T_0}{\rho_0}$.

Vibrating String - Separation of Variables

The **vibrating string** satisfies the following:

PDE: $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

BC: $$u(0,t) = 0, \quad u(L,t) = 0.$$

IC: $$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

This **vibrating string** problem or **wave equation** has fixed ends at $x = 0$ and $x = L$ and initial position, $f(x)$, and initial velocity, $g(x)$.

As before, we apply our **separation of variables** technique:

$$u(x,t) = \phi(x)h(t),$$

so

$$\phi'' h = c^2 \phi h'' \qquad \text{or} \qquad \frac{h''}{c^2 h} = \frac{\phi''}{\phi} = -\lambda.$$

Vibrating String - SL Problem

The ***homogeneous BCs*** give:

$$\phi(0) = 0 \qquad \text{and} \qquad \phi(L) = 0.$$

The **Sturm-Liouville Problem** becomes

$$\phi'' + \lambda\phi = 0 \qquad \text{with} \qquad \phi(0) = 0 = \phi(L).$$

As before, we saw $\lambda \le 0$ results in the ***trivial solution***.

If we take $\lambda = \alpha^2 > 0$, then

$$\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x),$$

where the **BCs** show $c_1 = 0$ and $\alpha = \frac{n\pi}{L}$ for nontrivial solutions.

The ***eigenvalues*** and ***associated eigenfunctions*** are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{with} \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Vibrating String - Superposition

The other ***second order DE*** becomes:

$$h'' + \frac{n^2\pi^2}{L^2}c^2h = 0,$$

which has the solution

$$h_n(t) = c_1\cos\left(\frac{n\pi ct}{L}\right) + c_2\sin\left(\frac{n\pi ct}{L}\right).$$

It follows that

$$u_n(x,t) = \left[A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

The **Superposition principle** gives:

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

Vibrating String - ICs

The **initial position** gives:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

The **velocity** satisfies

$$u_t(x,t) = \sum_{n=1}^{\infty}\left[-A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

The **initial velocity** gives:

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right),$$

where

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Physical Interpretation

Physical Interpretation: Model for vibrating string

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

- Musical instruments
- Each value of n gives a ***normal mode of vibration***
- **Intensity** depends on the ***amplitude***

$$A_n \cos(\omega t) + B_n \sin(\omega t) = \sqrt{A_n^2 + B_n^2} \sin(\omega t + \theta), \quad \theta = \arctan\left(\frac{A_n}{B_n}\right)$$

- Time dependence is ***simple harmonic*** with ***circular frequency***, $\frac{n\pi c}{L}$, which is the number of oscillations in 2π units of time
- The sound produced consists of superposition of the infinite number of ***natural frequencies***, $n = 1, 2, \ldots$

Physical Interpretation

Physical Interpretation (cont):

- The **normal mode**, $n = 1$, is called the ***first harmonic*** or ***fundamental mode***
- This mode has ***circular frequency***, $\frac{\pi c}{L}$
- Higher natural frequencies have higher pitch
- **Fundamental frequency** varied by changing, $c = \sqrt{\frac{T_0}{\rho_0}}$
  - Tune by changing tension, T_0
  - Different ρ_0 for different strings (range of notes)
  - Musician varies pitch by varying the length L (clamping string)
- Higher harmonics for stringed instruments are all integral multiples (pleasing to the ear)

Traveling Wave

Traveling Wave: Show that the solution to the vibrating string decomposes into two waves traveling in opposite directions.

- At each t, each mode looks like a simple oscillation in x, which is a ***standing wave***
- The amplitude simply varies in time
- The ***standing wave*** satisfies:

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\cos\left(\frac{n\pi}{L}(x-ct)\right) - \frac{1}{2}\cos\left(\frac{n\pi}{L}(x+ct)\right)$$

  - $\frac{1}{2}\cos\left(\frac{n\pi}{L}(x-ct)\right)$ produces a ***traveling wave*** to the right with velocity c
  - $\frac{1}{2}\cos\left(\frac{n\pi}{L}(x+ct)\right)$ produces a ***traveling wave*** to the left with velocity $-c$

- By **superposition** (later **d'Alembert's solution**)

$$u(x,t) = R(x-ct) + S(x+ct)$$