Math 3100

Section 1.5: Solution Sets of Linear Systems

Definition A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution $\mathbf{x} = \mathbf{0}$.

The solution $\mathbf{x} = \mathbf{0}$ is called the **trivial solution**. A more interesting question for a homogeneous system is

Does it have a nontrivial solution?

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a)
$$\begin{aligned} 2x_1 + x_2 &= 0 \\ x_1 - 3x_2 &= 0 \end{aligned}$$

In matrix form

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

There are two pivot columns, so there are no free variables.

The only solution is the trivial solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\text{(b)} \quad \begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

As a matrix equation

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using an augmented matrix with row reduction

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables and two pivot columns. So there is one free variable, hence nontrivial solutions.

We'll express the solution set using the rref. We will always express basic variables as functions of free variables (never visa versa)

From the rref

$$x_1 - \frac{4}{3}x_3 = 0$$

$$x_2 = 0$$

x_3 is free

$$x_1 = \frac{4}{3}x_3$$

$$x_2 = 0$$

x_3 - free

In vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

(c) $x_1 - 2x_2 + 5x_3 = 0$

As a matrix equation

$$\begin{bmatrix} 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & -2 & 5 & 0 \end{bmatrix}$ is already in rref.

There is one pivot column (the first) so there are two free variables.

$x_1 = 2x_2 - 5x_3$

x_2 is free

x_3 is free

In vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

x_2 and x_3 can be any real numbers.

Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $\mathbf{x} = x_3\mathbf{v}$. Example (c)'s solution set consisted of vector that look like $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$. Since these are **linear combinations**, we could write the solution sets like

$$\text{Span}\{\mathbf{v}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.$$

Instead of using the variables x_2 and/or x_3 we often substitute **parameters** such as s or t.
The forms

$$\mathbf{x} = s\mathbf{u}, \quad \text{or} \quad \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

are called **parametric vector forms.**

Example

The **parametric vector form** of the solution set of
$x_1 - 2x_2 + 5x_3 = 0$ is

is an element of

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

Question: What geometric object is that solution set?

This is a plane in $\mathbb{R}^3$ containing
$(0,0,0)$, $(2,1,0)$ and $(-5,0,1)$.

Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

we'll use an augmented matrix w/ row reduction.

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last column is not a pivot column. The system is consistent. There are 2 pivot columns and 3 variables, hence there is a free variable.

$$x_1 \quad -\frac{4}{3}x_3 = -1$$

$$x_2 \quad = 2$$

x_3 -free

as a vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 4/3\,x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3\,x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

This looks like the constant vector $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

plus any solution to the homogeneous equation with the same left hand side.

Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

with $\mathbf{p}$ and $\mathbf{v}$ fixed vectors and t a varying parameter. Also note that the $t\mathbf{v}$ part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

$\mathbf{p}$ is called a **particular solution**, and $t\mathbf{v}$ is called a solution to the associated homogeneous equation.

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

We can use a row reduction technique to get all parts of the solution in one process.