Calculus for the Biological Sciences

Lecture Notes – Limits, Continuity, and the Derivative

Ahmed Kaffel,
(ahmed.kaffel@marquette.edu)

Department of Mathematics and Statistics
Marquette University San
Milwaukee, WI 53233

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Outline

Introduction

- Limits are central to Calculus
- Present definitions of limits, continuity, and derivative
- Sketch the formal mathematics for these definitions
- Graphically show these ideas
- Recall derivative is related to the slope of the tangent line
- Complete understanding of the definitions is beyond the scope of this course

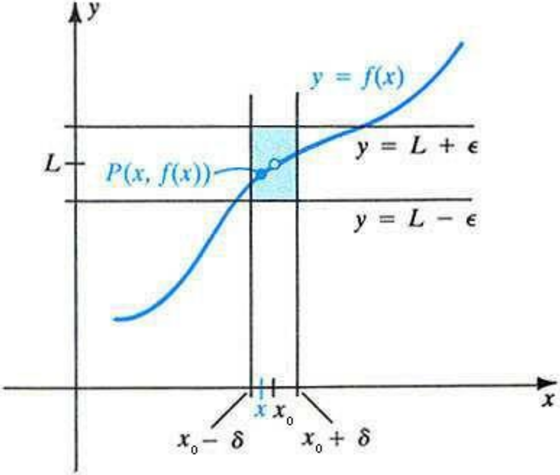
Definition of Limit

Limits –Conceptually, the limit of a function $f(x)$ at some point x_0 simply means that if your value of x is very close to the value x_0, then the function $f(x)$ stays very close to some particular value

Definition: The limit of a function $f(x)$ at some point x_0 exists and is equal to L if and only if every "small" interval about the limit L, say the interval $(L - \varrho, L + \varrho)$, means you can find a "small" interval about x_0, say the interval $(x_0 - \delta, x_0 + \delta)$, which has all values of $f(x)$ existing in the former "small" interval about the limit L, except possibly at x_0 itself

Definition of Limit

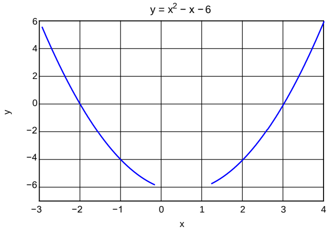
Diagram for Definition of Limit



Examples of Limits

Example of Limits: Consider $f(x) = x^2 - x - 6$

- Find the limit as x approaches 1
- From either the graph or from the way you have always evaluated this quadratic function that as x approaches 1, $f(x)$ approaches -6, since $f(1) = -6$

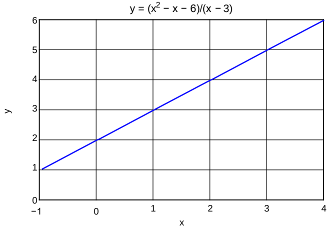


Fact: Any polynomial, $p(x)$, has as its limit at some x_0, the value of $p(x_0)$

Examples of Limits

Example of Limits: Consider $r(x) = \frac{x^2 - x - 6}{x - 3}$

- Find the limit as x approaches 1
- If x is not 3, then this rational function reduces to $r(x) = x + 2$
- So as x approaches 1, this function simply goes to 3

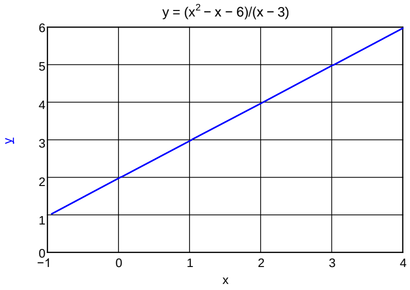


Fact: Any rational function, $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with $q(x_0)$ not zero, then the limit exists with the limit being $r(x_0)$

Examples of Limits

Example of Limits: Consider $r(x) = \frac{x^2 - x - 6}{x - 3}$

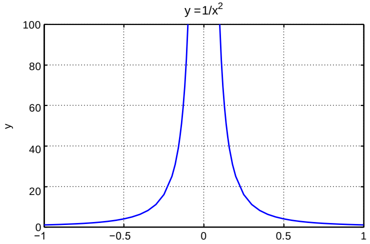
- Find the limit as x approaches 3
- Though $r(x)$ is not defined at $x_0 = 3$, arbitrarily "close" to 3, $r(x) = x + 2$
- So as x approaches 3, this function goes to 5
- Its limit exists though the function is not defined at $x_0 = 3$



Examples of Limits

Example of Limits: Consider $f(x) = \frac{1}{x^2}$

- Find the limit as x approaches 0, if it exists
- This function has a limit for any value of x_0 where the denominator is not zero
- However, at $x_0 = 0$, this function is undefined Thus,
- the graph has a vertical asymptote at $x_0 = 0$ This
- means that no limit exists for $f(x)$ at $x_0 = 0$



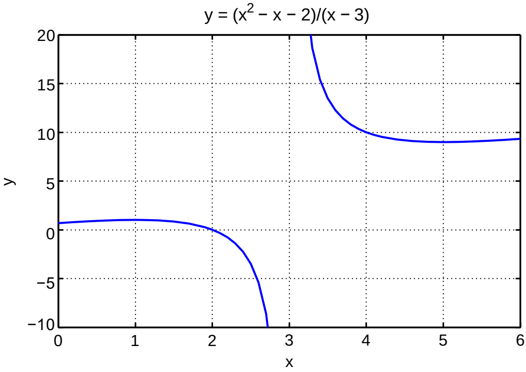
Examples of Limits

Example of Limits: Consider $r(x) = \frac{x^2 - x - 2}{x - 3}$

- Find the limit as x approaches 3, if it exists
- This function has a limit for any value of x_0 where the denominator is not zero
- Since the numerator is not zero, while the denominator is zero at $x_0 = 3$, this function is undefined at $x_0 = 3$
- The graph has a vertical asymptote at $x_0 = 3$ This
- means that no limit exists for $r(x)$ at $x_0 = 3$

Examples of Limits

For $r(x) = \frac{x^2 - x - 2}{x - 3}$



Fact: Whenever you have a vertical asymptote at some x_0, then the limit fails to exist at that point

Examples of Limits

Example of Limits: The Heaviside function is often used to specify when something is “on” or “off”

The Heaviside function is defined as

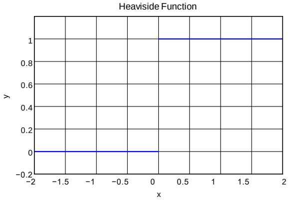
$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

- This function clearly has the limit of 0 for any $x < 0$, and it has the limit of 1 for any $x > 0$
- Even though this function is defined to be 1 at $x = 0$, it does not have a limit at $x_0 = 0$
  - If you take some “small” interval about the proposed limit of 1, say $\epsilon = 0.1$, then all values of x near 0 must have $H(x)$ between 0.9 and 1.1
  - But take any “small” negative x and $H(x) = 0$, which is not in the desired given interval
- Thus, no limit exists for $H(x)$

Examples of Limits

For

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

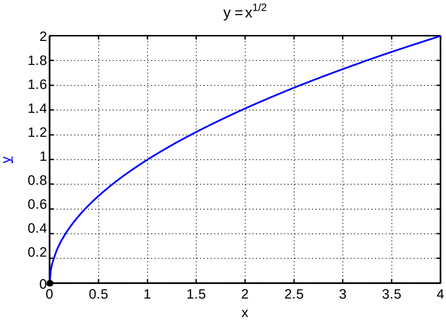


Perspective: Whenever a function is defined differently on different intervals (like the Heaviside function), check the x-values where the function changes in definition to see if the function has a limit at these x-values

Examples of Limits

Example of Limits: Consider $f(x) = \sqrt{x}$ −

- Find the limit as x approaches 0, if it exists
- This function is not defined for x < 0, so it cannot have a limit at x = 0, though it is said to have a right-handed limit



Summary of

Summary of Limits:

- Most of the functions in this course examine have limits
- Continuous portions of a function have limits
- Limits fail to exist at points x_0
    - At a vertical asymptote
    - When the function is defined differently on different intervals
    - Special cases like the square root function

Continuity

Continuity

- Closely connected to the concept of a limit is that of continuity
- Intuititvely, the idea of a continuous function is what you would expect
  - If you can draw the function without lifting your pencil, then the function is continuous
- Most practical examples use functions that are continuous or at most have a few points of discontinuity

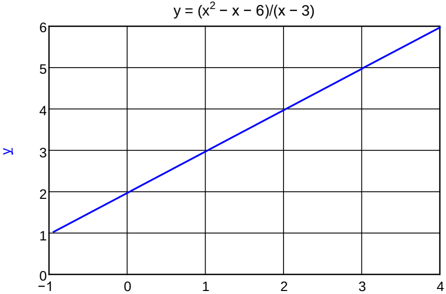
Definition: A function $f(x)$ is continuous at a point x_0 if the limit exists at x_0 and is equal to $f(x_0)$

Continuity in Examples

Example 3: For

$$r(x) = \frac{x^2 - x - 6}{x - 3}$$

- Though the limit exists at $x_0 = 3$, the function is not continuous there (function not defined at $x = 3$)
- This function is continuous at all other points, $x \neq 3$

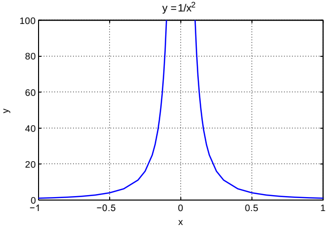


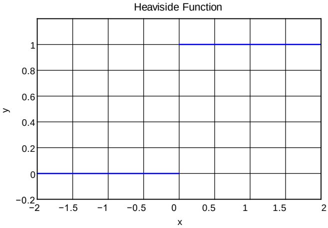
Continuity in Examples

Examples 4 and 6: For

$$f(x) = \frac{1}{x^2} \quad \text{and} \quad H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

- These functions are not continuous at $x_0 = 0$
- These functions are continuous at all other points, $x \neq 0$

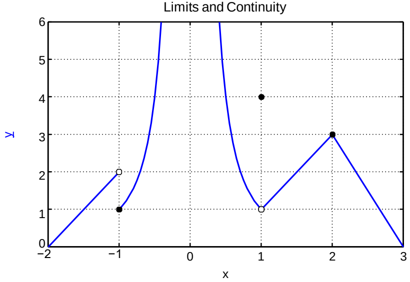




Comparing Limits and Continuity

Example:

Below is a graph of a function, $f(x)$, that is defined $x \in [-2,2]$, except at $x = 0$



Difficulties with this function occur at integer values

Comparing Limits and Continuity

At $x = -1$, the function has the value $f(-1) = 1$

The function is not continuous nor does a limit exist at this point At $x = 0$, the function is not defined

There is a vertical asymptote
At $x = 1$, the function has the value $f(1) = 4$

The function is not continuous, but the limit exists with

$$\lim_{x \to 1} f(x) = 1$$

At $x = 2$, the function is continuous with $f(2) = 3$, which also means that the limit exists
At all non-integer values of x the function is continuous (hence its limit exists)

We will see that the derivative only exists at these non-integer values of x

Derivative

Derivative

- The primary reason for the discussion above is for the proper definition of the derivative
- The derivative at a point on a curve is the slope of the tangent line at that point
- This motivation is what underlies the definition given below

Definition: The derivative of a function $f(x)$ at a point x_0 is denoted $f'(x_0)$ and satisfies

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists

Derivative of x^2

Example: Use the definition to find the derivative of

$$f(x) = x^2$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x
\end{aligned}$$

Derivative of $f(x) = 1/(x+2)$

Example: Use the definition to find the derivative of

$$f(x) = \frac{1}{x+2}, \qquad x \neq -2$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} \\
&= \lim_{h\to 0} \frac{(x+2)-(x+h+2)}{h(x+2+h)(x+2)} \\
&= \lim_{h\to 0} \frac{-h}{h(x+2+h)(x+2)} \\
&= \lim_{h\to 0} \frac{-1}{(x+2+h)(x+2)} \\
&= \frac{-1}{(x+2)^2}
\end{aligned}
$$

Derivatives

- Clearly, we do not want to use this formula every time we need to compute a derivative
- Much of the remainder of this course will be learning easier ways to take the derivative
- In Lab, a very easy way to find derivatives is using the Maple diff command