Section 1.1: Systems of Linear Equations

We begin with a linear (*algebraic*) equation in n variables x_1, x_2, ..., x_n for some positive integer n.

A **linear equation** can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

The numbers $a_1, \ldots, a_n$ are called the *coefficients*. These numbers and the right side b are real (or complex) constants that are **known.**

Questions

- Is there a set of numbers $x_1, \ldots, x_4$ that satisfy all of the equations?

  Existence question

- If there is a set of numbers, is it the only one?

  uniqueness question

Examples of Equations that are or are not Linear

Both Linear

$$2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x+y) = 0$$

$$2x_1 - 4x_2 + 3x_3 = 5$$

$$-\sqrt{3}x - \sqrt{3}y = -12$$

$$x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w}$$

Both not linear

nonlinear divide by variable

nonlinear product of variables

A *Linear System* is a collection of linear equations in the same variables

$$
\begin{array}{rcrcrcrcr}
2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\
-x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8
\end{array}
$$

$$
\begin{array}{rcrcrcr}
x & + & 2y & + & 3z & = & 4 \\
3x & + & 0y & + & 12z & = & 0 \\
2x & + & 2y & - & 5z & = & -6
\end{array}
$$

Some terms

ordered n-tuple

- A **solution** is a list of numbers $(s_1, s_2, \ldots, s_n)$ that reduce each equation in the system to a true statement upon substitution.
- A **solutions set** is the set of all possible solutions of a linear system.
- Two systems are called **equivalent** if they have the same solution set.

An Example

$$\begin{array}{rcrcr} 2x & - & y & = & -1 \\ -4x & + & 2y & = & 2 \end{array}$$

(a) Show that $(1, 3)$ is a solution.

Set $x=1$ and $y=3$. We have to show that both equations reduce to identities.

equation 1 :

$$2(1) - (3) \stackrel{?}{=} -1$$

$$2 - 3 = -1$$

$$-1 = -1$$

equation 2:

$$-4(1) + 2(3) \stackrel{?}{=} 2$$

$$-4 + 6 = 2$$

$$2 = 2$$

(1, 3) is a solution.

An Example Continued

$$\begin{array}{rcrcr} 2x & - & y & = & -1 \\ -4x & + & 2y & = & 2 \end{array}$$

(b) Note that $\{(x, y) | y = 2x + 1\}$ is the solution set.

Let's show both reduce to identities:

equation 1 $\quad 2x - (2x+1) \stackrel{?}{=} -1$

$2x - 2x - 1 = -1$

$-1 = -1$

equation 2 $\quad -4x + 2(2x+1) \stackrel{?}{=} 2$

$-4x + 4x + 2 = 2$

$$2 = 2$$

From equation 1:

$$2x - y = -1$$

add y and 1 to both sides

$$2x + 1 = y$$