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ON METRIC-RELATED BETWEENNESS

PAUL BANKSTON, AISLING MCCLUSKEY, AND W. STEPHEN WATSON

ABSTRACT. We study betweenness relations and their first-order axiomatizations arising in connection with metric spaces. This includes the usual metric betweenness of Karl Menger [11], as well as versions of betweenness that make sense in light of generalized triangle (in)equalities. In particular, there is an ultrametric version whose betweenness behavior strongly resembles that seen in the subcontinuum betweenness of hereditarily indecomposable continua.

1. INTRODUCTION AND PRELIMINARIES

A **ternary structure** is a relational structure $\langle X, I \rangle$, where X is a nonempty set and $I = I(\cdot, \cdot, \cdot)$ is a ternary relation on X . Officially we read $I(a, c, b)$ as “ c lies between a and b .” To reflect this intuition we define a ternary structure to be a **betweenness structure** if it satisfies the following (universally quantified) **basic betweenness axioms**, expressed in the first-order language with one ternary relation symbol (as well as the binary equality symbol).

- (*Inclusivity*) $I(u, v, v) \wedge I(u, u, v)$
- (*Symmetry*) $I(u, w, v) \rightarrow I(v, w, u)$
- (*Uniqueness*) $I(u, v, u) \rightarrow u = v$.

We abuse notation slightly and write, for each $a, b \in X$, the **I -interval**

$$I(a, b) := \{x \in X : I(a, x, b) \text{ holds}\}.$$

In interval terms, inclusivity says that $\{a, b\} \subseteq I(a, b)$; symmetry says $I(a, b) = I(b, a)$, and uniqueness says $I(a, a) = \{a\}$. The set $\{a, b\}$ is called a **bracket set** for the interval; in general an interval may have many bracket sets, so we avoid using the term *end points*. An interval—or any set—is **nondegenerate** if it contains at least two points; a nondegenerate interval containing just two points is called a **gap**.

Classically, a betweenness relation on a set arises naturally from some *antecedent* structure on the set. Such antecedents typically include orderings, graphs, vector operations, metrics, and topologies. In [6] we introduce a new antecedent, which we call a **road system**. This is just a family $\mathcal{R}$ of nonempty subsets of the set X —called *roads*—such that: (1) $X \in \mathcal{R}$; and (2) $\{a\} \in \mathcal{R}$ for each $a \in X$. A road system $\mathcal{R}$ **induces** a ternary relation I by defining

$$I(a, b) := \bigcap \{R \in \mathcal{R} : \{a, b\} \subseteq R\}.$$

Informally, c lies $\mathcal{R}$ -between a and b if every road containing $\{a, b\}$ contains c as well. A given ternary relation is **road-induced** if some road system induces it.

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Clearly road-induced ternary structures satisfy the basic axioms above, and are therefore betweenness structures in our sense.

1.1. Remark. In [3] it is pointed out that the two defining conditions for $\mathcal{R}$ to be a road system on X may be replaced by the single *Fréchet condition*—analogous with the T_1 axiom in topology—that each of any two points of X lies in a road that excludes the other. Any road system in the old sense is clearly a road system in the new; and by adding X and all singletons to a road system in the new sense we obtain a road system in the old, one that induces the same betweenness structure.

In this paper the term *metric* is used in its classical sense. That is, a **metric** on a set X is a function $\varrho : X^2 \rightarrow [0, \infty) \subseteq \mathbb{R}$ such that the following conditions universally hold: (*positivity*) $\varrho(a, b) = 0$ if and only if $a = b$; (*metric symmetry*) $\varrho(a, b) = \varrho(b, a)$; and (*triangle inequality*) $\varrho(a, b) \leq \varrho(a, c) + \varrho(c, b)$. A set $V \subseteq [0, \infty)$ is a **value set** for ϱ if $\varrho(a, b) \in V$ for all $a, b \in X$. We then say the metric is **V -valued**.

The metric ϱ can induce a ternary relation on X in a number of ways; the most well known of these is to stipulate the following *triangle equality* condition:

$$I(a, c, b) \text{ if and only if } \varrho(a, b) = \varrho(a, c) + \varrho(c, b).$$

This is evidently a betweenness relation, and was first studied by Karl Menger in [11].¹ We call a ternary relation induced by a metric in this way **$+$ -induced**.² This version of betweenness also satisfies the following two conditions.

- (*Transitivity*) $(I(u, s, v) \wedge I(u, t, s)) \rightarrow I(u, t, v)$
- (*Reversal*) $(I(u, s, v) \wedge I(u, t, s)) \rightarrow I(t, s, v)$.

Formally weaker than reversal is the following.

- (*Inversion*)³ $(I(u, s, v) \wedge I(u, t, v) \wedge I(u, t, s)) \rightarrow I(t, s, v)$.

Betweenness structures satisfying transitivity and inversion are called *geometric* in [16]. Clearly reversal and inversion are equivalent for betweenness structures satisfying transitivity.

Transitivity for betweenness structures is related to usual transitivity for binary structures because for each $a \in X$, we may define the binary relation $\leq_a$ on X by the condition that

$$x \leq_a y \text{ if and only if } x \in I(a, y).$$

Each *slice* $\leq_a$ is reflexive because of inclusivity, and is transitive (in the binary sense) because of transitivity. This makes $\leq_a$ into a pre-order. And because of the uniqueness axiom, a is its unique least element.

The proofs of transitivity and reversal for $+$ -induced ternary relations are accomplished simultaneously in [11]. Indeed, suppose that $a, b, c, d \in X$ are such that both $I(a, c, b)$ and $I(a, d, c)$ hold for the $+$ -induced betweenness relation. We show that $I(a, d, b)$ and $I(d, c, b)$ both hold. By the triangle inequality, we know that the two equalities

$$(1) \varrho(a, d) + \varrho(d, b) = \varrho(a, b) + \alpha$$

¹Menger actually stipulated that $c \notin \{a, b\}$ as well—*exclusivity*—but the two approaches are essentially the same.

²We later consider operations other than addition in ways of defining metric-induced betweenness.

³The terminology is from [16]; also called *reciprocity* in [6].

$$(2) \varrho(d, c) + \varrho(c, b) = \varrho(d, b) + \beta$$

hold for some $\alpha, \beta \geq 0$; we wish to show $\alpha = \beta = 0$. By the first clause of the premise, we have

$$(3) \varrho(a, b) = \varrho(a, c) + \varrho(c, b),$$

so substitution of (3) in (1) gives

$$(4) \varrho(a, d) + \varrho(d, b) = \varrho(a, c) + \varrho(c, b) + \alpha.$$

But by the second clause of the premise, we have

$$(5) \varrho(a, c) = \varrho(a, d) + \varrho(d, c).$$

Substituting (5) in (4) gives us

$$(6) \varrho(a, d) + \varrho(d, b) = \varrho(a, d) + (\varrho(d, c) + \varrho(c, b)) + \alpha.$$

Finally, apply (2) to the sum within parentheses to obtain

$$\varrho(a, d) + \varrho(d, b) = \varrho(a, d) + \varrho(d, b) + \alpha + \beta,$$

and hence conclude that $\alpha + \beta = 0$. Both these numbers are nonnegative, and are therefore zero.

Any betweenness structure satisfying reversal also satisfies

$$\bullet \text{ (Antisymmetry) } (I(u, w, v) \wedge I(u, v, w)) \rightarrow v = w.$$

(To see this, apply reversal to the premise of antisymmetry to conclude $I(v, w, v)$; then apply uniqueness.) In a betweenness structure satisfying antisymmetry, any two bracket sets for the same interval must be disjoint because if $I(a, b) = I(a, c)$, then both $I(a, c, b)$ and $I(a, b, c)$ hold, and so $b = c$.

When we translate antisymmetry to the binary relations $\leq_a$ introduced above, this axiom takes the more familiar form of asserting that each of these relations is antisymmetric in the usual binary sense. Hence in betweenness structures satisfying transitivity and antisymmetry, each relation $\leq_a$ is a partial ordering, with unique least element a .

So let $\langle X, I \rangle$ be a betweenness structure satisfying both transitivity and antisymmetry. For $a \in X$ and $C \subseteq X$, we say that C is an **a -chain** if C is totally ordered by $\leq_a$. A subset of X is a **chain** if it is an a -chain for some $a \in X$. In particular it makes sense to talk about chains in betweenness structures that are $+$ -induced. We denote by $|A|$ the cardinality of a set A ; if A is a chain, we refer to its cardinality as its **length**. The following is a crude, but useful, condition necessary for a betweenness relation to be $+$ -induced by a metric.

1.2. Proposition. *Suppose $\langle X, \varrho \rangle$ is a metric space, with I the betweenness relation $+$ -induced by ϱ . Then every chain of $\langle X, I \rangle$ has length at most $|V| \leq \mathfrak{c}$, where V is a value set for ϱ and $\mathfrak{c} := |\mathbb{R}| = 2^{\aleph_0}$.*

Proof. Let V be a value set for ϱ , and $C \subseteq X$ an a -chain for some $a \in X$. If $b, c \in C$ are distinct, then either $c \in I(a, b)$ or $b \in I(a, c)$; say $c \in I(a, b)$. Hence we have that $\varrho(a, b) = \varrho(a, c) + \varrho(c, b)$, which implies $\varrho(a, c) < \varrho(a, b)$. This tells us that the function $f : C \rightarrow V$, given by $f(x) = \varrho(a, x)$, is injective; hence $|C| \leq |V| \leq \mathfrak{c}$. $\square$

We will have more to say about a -chains in Sections 4 and 7.

Not only are road-induced ternary relations betweenness relations, they also satisfy the following strong version of transitivity.

$$\bullet \text{ (Convexity) } (I(u, s, v) \wedge I(u, t, v) \wedge I(s, w, t)) \rightarrow I(u, w, v).$$

This is a key betweenness property, as is demonstrated by [6, Theorem 2.0.1] and [3, Theorem 3.3].

1.3. Theorem. *A betweenness structure is road-induced if and only if it satisfies convexity.*

1.4. Remark. In model-theoretic parlance [10], Theorem 1.3 establishes that the class of road-induced ternary relational structures is an *elementary class* that is axiomatizable via a finite set of universal sentences.

Let $\langle X, I \rangle$ be a betweenness structure, with $B \subseteq A \subseteq X$. Then A is ***I*-star-shaped about B** if whenever $a, b \in A$, with $b \in B$, we have $I(a, b) \subseteq A$. Another paraphrase of transitivity is that I -intervals are I -star-shaped about any of their bracket sets. A set $A \subseteq X$ is ***I*-convex** if it is I -star-shaped about itself. Convexity then says that I -intervals are themselves I -convex.

There are copious situations where $+$ -induced betweenness structures fail to satisfy convexity (see, e.g., [3, 11]), and so we conclude that such betweenness structures are not generally road-induced.

A second—but less well known—way to consider a ternary relation to be induced by a metric is to stipulate the following *strong triangle equality* condition:

$$I(a, c, b) \text{ if and only if } \varrho(a, b) = \varrho(a, c) \sqcup \varrho(c, b),$$

(where $x \sqcup y := \max\{x, y\}$ for real numbers). This is also a betweenness relation, and we call a ternary relation induced by a metric in this way **$\sqcup$ -induced**. For clarity, we indicate with superscripts these two betweenness relations induced by a metric ϱ ; namely $I^+ = I_\varrho^+$ and $I^\sqcup = I_\varrho^\sqcup$ denote the $+$ -induced and the $\sqcup$ -induced betweenness relations/intervals, respectively.

We will see that $\sqcup$ -induced betweenness structures are most interesting when the inducing metric is an **ultrametric**; ⁴ i.e., when it satisfies the *strong triangle inequality*

$$\varrho(x, y) \leq \varrho(x, z) \sqcup \varrho(z, y).$$

We compare the betweenness relations $+$ -induced and $\sqcup$ -induced by a given metric in the next section. In Sections 3 and 4 we examine betweenness relations $+$ -induced and $\sqcup$ -induced, respectively, by metrics with finite value sets. In those sections we prove some characterization theorems, and in Section 5 we examine those characterizations in terms of “omitted patterns,” reminiscent of K. Kuratowski’s characterization of finite simple graphs that are embeddable in the Euclidean plane. In Section 6 we generalize the notion of *triangle (in)equality* to create a spectrum of metric-induced betweenness notions with $\sqcup$ and $+$ at the extremes. In Section 7 we consider when—and to what extent—a betweenness relation is *compatible* with a metric, and obtain results having to do with the existence of metrics compatible with a given betweenness relation. In Section 8, we relate the betweenness $\sqcup$ -induced by an ultrametric and that induced by the road system of subcontinua of a hereditarily indecomposable continuum. In particular, we show that any metric generating the topology of a hereditarily indecomposable continuum naturally refines to an ultrametric whose $\sqcup$ -induced betweenness relation contains the subcontinuum betweenness relation. Finally, in Section 8 we present a short list of open questions.

⁴Also called *non-Archimedean* metrics, ultrametries have applications in p -adic analysis, physics, and computer science, among others. Our interest is purely geometric.

2. THE TWO METRIC BETWEENNESS RELATIONS COMPARED

First we show the two relations have only a minimal amount in common.

2.1. Proposition. *In any metric space $\langle X, \varrho \rangle$, and any two points $a, b \in X$, we have $I^+(a, b) \cap I^\sqcup(a, b) = \{a, b\}$. Moreover, if the inducing metric is an ultrametric then $I^+(a, b) = \{a, b\}$.*

Proof. If $c \notin \{a, b\}$, then $\varrho(a, c) + \varrho(c, b) > \varrho(a, c) \sqcup \varrho(c, b)$. Hence no point other than a bracket point can lie in both intervals. The second assertion is immediate from the definition of *ultrametric*. $\square$

By way of notation: for a metric space $\langle X, \varrho \rangle$, $c \in X$, and $r \geq 0$, define $B(c; r) = B_\varrho(c; r)$ (resp., $B^\circ(c; r) = B_\varrho^\circ(c; r)$, $S(c; r) = S_\varrho(c; r)$) to be $\{x \in X : \varrho(c, x) \leq r\}$ (resp., $\{x \in X : \varrho(c, x) < r\}$, $\{x \in X : \varrho(c, x) = r\}$). These are respectively the *closed ball*, *open ball*, and *sphere*, associated with ϱ , that has *center* c and *radius* r . In practice all radii are assumed positive; but the following is true even when $r = 0$, and is an easy consequence of the definition.

2.2. Lemma. *Let $\langle X, \varrho \rangle$ be a metric space, $a, b \in X$, with $r = \varrho(a, b)$. Then*

$$I^\sqcup(a, b) = (S(a; r) \cap B(b; r)) \cup (B(a; r) \cap S(b; r)).$$

2.3. Examples. (i) Consider $X = \mathbb{R}_2^2$, the Cartesian plane as a normed real vector space with the Euclidean norm. Pick $a, b \in X$ distinct. Then $I^+(a, b) = \{(1-t)a + tb : 0 \leq t \leq 1\}$, the usual line segment with a and b as end points. But by Lemma 2.2, $I^\sqcup(a, b)$ is a lens-shaped simple closed curve, which is the union of two circular arcs, each sweeping out an angle of $\frac{2\pi}{3}$ radians. If $c \in I^\sqcup(a, b) \setminus \{a, b\}$, then it cannot be that $I^\sqcup(a, c) \subseteq I^\sqcup(a, b)$; hence transitivity fails quite dramatically for $\sqcup$ -induced betweenness in this case. The two circular arcs constituting $I^\sqcup(a, b)$ intersect in two cusp points. If c is one of these, then we have $c \in I^\sqcup(a, b)$ and $b \in I^\sqcup(a, c)$, while $b \neq c$. Thus antisymmetry fails in this case too.

(ii) Indeed, if X is any normed plane; i.e., the Cartesian plane $\mathbb{R}^2$ as a real vector space equipped with a norm $\|\cdot\|$, then [4, Theorem 5.9] shows that when $a, b \in X$ are distinct, each $I^+(a, b)$ is either a line segment with end points a and b , or a solid parallelogram with a and b at opposite corners. On the other hand, if $r = \|a - b\|$, then $B(a; r) \cap B(b; r)$ is a compact linearly convex set with the midpoint $\frac{1}{2}a + \frac{1}{2}b$ in its interior. By Lemma 2.2, $I^\sqcup(a, b)$ is the topological boundary of $B(a; r) \cap B(b; r)$, which, by classical plane convexity theory, is a simple closed curve.

For example, when $\|\cdot\|$ is the *taxicab norm* (alias, *Manhattan norm*, *rook's norm*), $I^+(a, b)$ is a solid rectangle when a and b do not share any rectangular coordinates; and $I^\sqcup(a, b)$ is a hollow parallelogram with sides parallel to the diagonal lines $y = \pm x$.

(iii) In this paper, all graphs are assumed to be *simple*; i.e., no loops and no more than one edge between any two distinct points. Any metric space $\langle X, \varrho \rangle$ may be viewed as an edge-labeled complete graph, where $\varrho(a, b)$ is a label for the *edge* determined by distinct vertices $a, b \in X$. Any complete graph with positive labels for its edges may be viewed as a metric space, as long as the labels for each triangle obey the triangle inequality. (To view the graph as an *ultrametric* space, the labels for each triangle must

obey the *strong* triangle inequality.) We call a triangle in such a complete graph **isosceles** if two of its edges have equal labels. An isosceles triangle is **tall** (resp., **short**) if the two equal edges have labels larger (resp., smaller) than the third; otherwise the isosceles triangle is termed **equilateral**. The common vertex of the two equal edges of an isosceles triangle that is either tall or short is the **apex** of the triangle.

For example, when the label set is $\{1, 2\}$, each triangle is isosceles. If a, b, c are distinct points, then $I^+(a, c, b)$ holds precisely when the corresponding triangle is short isosceles, with apex c . On the other hand, $I^\sqcup(a, c, b)$ holds precisely when the triangle is either equilateral or tall isosceles, with c not the apex. A triangle in a metric space is **+trivial** (resp., **$\sqcup$ -trivial**) if none of its three edge labels is equal to the sum (resp., maximum) of the other two. No triangle can be both $+$ -nontrivial and $\sqcup$ -nontrivial for any given metric. If the metric is an ultrametric, all triangles—see Lemma 2.4 below—are $\sqcup$ -nontrivial, and hence $+$ -trivial. When the label set is $\{1, 2\}$, the $+$ -nontrivial triangles are exactly the short isosceles ones. (We will have more to say in this connection in Sections 3 and 4.)

By Proposition 2.1, any betweenness relation that is $+$ -induced by an ultrametric is *trivial*, in the sense that all nondegenerate intervals are gaps. We will see that betweenness relations that are $\sqcup$ -induced by an ultrametric are quite nontrivial; they do satisfy transitivity (and much more), but almost never satisfy antisymmetry (see Theorems 2.10, 2.12, and Proposition 2.15 below).

2.4. Lemma. *Let $\langle X, \varrho \rangle$ be an ultrametric space, with $a, b, c \in X$ such that $c \notin I^\sqcup(a, b)$. Then $\varrho(a, c) = \varrho(c, b) > \varrho(a, b)$. (Hence $a \in I^\sqcup(b, c)$ and $b \in I^\sqcup(a, c)$.)*

Proof. By the hypothesis and the strong triangle inequality, we have $\varrho(a, c) \sqcup \varrho(c, b) > \varrho(a, b)$. Assume $\varrho(a, c) \leq \varrho(c, b)$. Then $\varrho(a, b) < \varrho(c, b)$. But then we also have $\varrho(c, b) \leq \varrho(a, b) \sqcup \varrho(a, c)$. Hence we have $\varrho(c, b) \leq \varrho(a, c)$, giving us equality. The parenthetical conclusion is immediate from the definition. $\square$

An easy geometric consequence of Lemma 2.4 is the well-known fact that in an ultrametric space, each triple a, b, c of distinct points forms the vertex set of a triangle that is either equilateral or tall isosceles. In the first instance, $I^\sqcup(a, c, b)$ automatically holds; in the second it holds if and only if c is not the apex of the triangle. Ultrametric spaces therefore contain no $\sqcup$ -trivial triangles (as mentioned above).

In a metric space $\langle X, \varrho \rangle$, with $a, b \in X$,

$$E(a, b) := \{x \in X : \varrho(a, x) = \varrho(x, b)\}$$

denotes the **equiset with cocenters** a, b . Let $M^+(a, b) := I^+(a, b) \cap E(a, b)$ and $M^\sqcup(a, b) := I^\sqcup(a, b) \cap E(a, b)$ be the associated **+midset** and **$\sqcup$ -midset**, respectively. (Equisets and $+$ -midsets are studied extensively in [4] and elsewhere; $\sqcup$ -midsets seem to be new.) In Example 2.3 (i), when a, b are distinct, we have $M^+(a, b) = \{\frac{1}{2}a + \frac{1}{2}b\}$, the usual midpoint of $I^+(a, b)$, while $M^\sqcup(a, b)$ comprises the two cusp points of $I^\sqcup(a, b)$. In an ultrametric space, $M^+(a, b)$ is empty when $a \neq b$.

The following is an easy consequence of Lemma 2.4.

2.5. Proposition. *Let $\langle X, \varrho \rangle$ be an ultrametric space, with $a, b \in X$. Then $I^\sqcup(a, b) \cup E(a, b) = X$ and $M^\sqcup(a, b) = \{x \in X : \varrho(x, a) = \varrho(x, b) = \varrho(a, b)\}$.*

Clearly $M^\sqcup(a, a) = \{a\} \cap E(a, a) = \{a\} \cap X = \{a\}$. But if $a \neq b$, then $M^\sqcup(a, b)$ is nonempty if and only if there is an equilateral triangle with $\{a, b\}$ constituting two of its vertices. Any two-valued metric on X —i.e., one with only a single non-zero value—is easily seen to be an ultrametric; in that situation, if $a, b \in X$ are distinct, then $M^\sqcup(a, b) = X \setminus \{a, b\}$.

2.6. Proposition. *Let $\langle X, \varrho \rangle$ be an ultrametric space. Then ϱ is two-valued if and only if all triangles in X are equilateral.*

Proof. If ϱ is two-valued, then there is just one label; hence all triangles are equilateral. If ϱ is not two-valued, then we have $a, b, c, d \in X$ where $0 < \varrho(a, b) < \varrho(c, d)$. If, say, $c \in \{a, b\}$, then $d \notin \{a, b\}$, and the triangle with vertices $\{a, b, d\}$ is tall. If $\{a, b\} \cap \{c, d\} = \emptyset$ and $\varrho(a, c) \neq \varrho(a, b)$, then the triangle $\{a, b, c\}$ is tall. Otherwise, the triangle $\{a, c, d\}$ is tall. $\square$

2.7. Example. An ultrametric example less trivial than the two-valued one is to let $X = S^\omega$, all infinite sequences of elements from set S ; and, for $a \neq b$, to define $\varrho(a, b) := 2^{-n}$, where n is the least $m < \omega$ such that $a(m) \neq b(m)$. When S has just two points, there are no equilateral triangles; and hence each midset $M^\sqcup(a, b)$ is either $\{a\}$, if $a = b$, or is empty.

In an arbitrary metric space $\langle X, \varrho \rangle$ let $\mathcal{B} = \mathcal{B}_\varrho$ (resp., $\mathcal{B}^\circ = \mathcal{B}_\varrho^\circ$) be the family of all closed (resp., open) ϱ -balls of X . From a topological perspective, $\mathcal{B}^\circ$ constitutes an open base for the metric topology.

Define a family $\mathcal{F}$ of subsets of X to be **rank one** if any two sets in $\mathcal{F}$ are either disjoint or nested.⁵ In the ultrametric case, the following is well known.

2.8. Proposition. *Let $\langle X, \varrho \rangle$ be a metric space. If ϱ is an ultrametric then both $\mathcal{B}$ and $\mathcal{B}^\circ$ are rank one families. Conversely, if either $\mathcal{B}$ or $\mathcal{B}^\circ$ is a rank one family, then ϱ is an ultrametric.*

Proof. Assume that ϱ is an ultrametric, and suppose $c \in B(a; r) \cap B(b; s)$, say $s \leq r$. Then we have

- (1) $\varrho(a, c) \leq r$ and $\varrho(b, c) \leq r$; hence
- (2) $\varrho(a, b) \leq r$.

Suppose $x \in B(b; r)$; i.e., $\varrho(b, x) \leq r$. By (2) we also have $\varrho(a, x) \leq r$; so $x \in B(a; r)$. This shows $B(a; r) = B(b; r)$. Since $s \leq r$, we have $B(b; s) \subseteq B(a; r)$, and conclude that $\mathcal{B}$ is a rank one family. By a similar argument, we conclude that $\mathcal{B}^\circ$ is also rank one.

If ϱ is not an ultrametric, then we have $a, b, c \in X$ with

$$\varrho(a, b) > r = \varrho(a, c) \sqcup \varrho(c, b).$$

Then $a \notin B(b; r)$ and $b \notin B(a; r)$, but $c \in B(a; r) \cap B(b; r)$. Hence $\mathcal{B}$ is not a rank one family. Similarly, we show $\mathcal{B}^\circ$ is not rank one either. $\square$

⁵For $0 < n < \omega$, J. Nagata [13] defines a family $\mathcal{F}$ of subsets of X to be of *rank n* if each subfamily consisting of at least $n + 1$ sets contains two sets that are either disjoint or nested.

2.9. Remark. In the ultrametric case, a consequence of Proposition 2.8 is that each member of $\mathcal{B}^\circ$ is closed as well as open (though not necessarily equal to its corresponding closed ball; witness the $\{0, 1\}$ -valued metric). As a result, all ultrametric spaces are topologically zero-dimensional.

More germane to the present discussion is the following.

2.10. Theorem. *If $\langle X, \varrho \rangle$ is an ultrametric space, then the betweenness relation on X $\sqcup$ -induced by ϱ is induced by the rank one road system $\mathcal{B}_\varrho$.*

Proof. Let $a, b \in X$. If $a = b$ then $I^\sqcup(a, b) = \{a\} = B(a; 0)$. Suppose now $r = \varrho(a, b) > 0$. If $x \in I^\sqcup(a, b)$ then clearly $\varrho(a, x) \leq r$; hence $I^\sqcup(a, b) \subseteq B(a; r)$. If $x \notin I^\sqcup(a, b)$, then, by Lemma 2.4, we have $\varrho(a, x) = \varrho(b, x) > r$; hence $x \notin B(a; r)$. This shows that $I^\sqcup(a, b) = B(a; r) = B(b; r)$. Thus not only is the $\sqcup$ -induced betweenness relation road-induced by $\mathcal{B}$, the family of $\sqcup$ -induced intervals coincides with $\mathcal{B}$. $\square$

Thus we see that, by Theorem 1.3, $\sqcup$ -induced betweenness in ultrametric spaces satisfies convexity, not just transitivity.⁶ But there is more.

- (*Disjunctivity*) $I(u, w, v) \rightarrow (I(u, w, t) \vee I(v, w, t))$.

In interval terms, disjunctivity says that $I(a, b) \subseteq I(a, c) \cup I(c, b)$ for all a, b, c . If we restrict ourselves to the additional hypothesis that $c \in I(a, b)$, we have so-called *weak disjunctivity*. In the presence of transitivity, weak disjunctivity says that $I(a, b) = I(a, c) \cup I(c, b)$ whenever $c \in I(a, b)$. It is easy to show that this in turn implies convexity (see [6, Theorem 4.0.1]). Also in betweenness structures satisfying both weak disjunctivity and transitivity, antisymmetry is a sufficient—as well as necessary—condition for inversion [6, Theorem 5.0.6] (for reversal as well).

A road system is **additive** if the union of two overlapping roads is also a road. The following accompaniment to Theorem 1.3 is a slight reworking of [6, Theorem 4.0.5].

2.11. Theorem. *A betweenness structure is induced by an additive road system if and only if it satisfies transitivity and disjunctivity.*

Proof. First suppose $\langle X, I \rangle$ is a betweenness structure induced by an additive road system. The fact that it satisfies both transitivity—even convexity—and disjunctivity is straightforward to prove (see [6, Theorem 4.0.5]).

For the converse it suffices—in light of [6, Theorem 4.0.5]—to show that if $\langle X, I \rangle$ is a betweenness structure satisfying both transitivity and disjunctivity, then it also satisfies convexity. Indeed, by [6, Theorem 4.0.1], all we need is weak disjunctivity and transitivity to infer convexity. $\square$

Since rank one road systems are trivially additive, we conclude that $\sqcup$ -induced betweenness in ultrametric spaces satisfies disjunctivity; but we can go further. Consider the following betweenness condition.

- (*Totality*) $I(u, w, v) \vee I(u, v, w)$.

In the indexed binary notation introduced above, this says that each relation $\leq_a$ is *total*, in the binary sense that any two elements of the domain are $\leq_a$ -comparable. We can now add to Theorems 1.3 and 2.11.

⁶Note how different are the proofs of transitivity for $+$ -induced and $\sqcup$ -induced betweenness relations.

2.12. Theorem. *A betweenness structure is induced by a rank one road system if and only if it satisfies transitivity and totality.*

Proof. Let $\mathcal{R}$ be a rank one road system inducing the betweenness structure $\langle X, I \rangle$. Transitivity—even convexity—is straightforward. As for totality, suppose $a, b, c \in X$ and it is not the case that $I(a, c, b)$ holds. Then there is some $R \in \mathcal{R}$ such that $\{a, b\} \subseteq R$ but $c \notin R$. Let $S \in \mathcal{R}$ be any road containing $\{a, c\}$. Since $a \in R \cap S$ and $S \not\subseteq R$ (as witnessed by c), we conclude that $R \subseteq S$, by the rank one condition. Thus $b \in S$, showing that $I(a, b, c)$ holds; hence our betweenness structure satisfies totality.

Suppose $\langle X, I \rangle$ is a betweenness structure satisfying transitivity and totality. We first show that convexity holds. (The simple proof is similar to that of [6, Theorem 4.0.1], where *weak disjunctivity* replaces *totality*.) Assume $a, b, c, d, e \in X$ are such that $I(a, c, b)$, $I(a, d, b)$, and $I(c, e, d)$ all hold; we wish to show $I(a, e, b)$ holds. By totality, we know at least one of $I(a, c, d)$, $I(a, d, c)$ holds, say it is the first one. Then $I(a, e, d)$ holds, by one application of transitivity, and $I(a, e, b)$ holds by another.

And now, since $\langle X, I \rangle$ satisfies convexity, I is induced by a road system $\mathcal{R}$, by Theorem 1.3. Assuming $\mathcal{R}$ is not rank one, there are two roads $R, S \in \mathcal{R}$ that overlap, but are not nested. Let $a \in R \cap S$, $b \in R \setminus S$, and $c \in S \setminus R$ be witness to this fact. Then neither $I(a, c, b)$ nor $I(a, b, c)$ holds, contradicting totality. $\square$

- 2.13. Remarks.**
- (i) By Theorems 2.11 and 2.12, we infer that betweenness structures satisfying transitivity and totality also satisfy disjunctivity.
 - (ii) Again in model-theoretic terms, Theorem 2.11 (resp., Theorem 2.12) establishes that the class of ternary relational structures that are induced by an additive (resp., a rank one) road system is an elementary class, which is axiomatizable via a finite set of universal sentences.
 - (iii) If $\langle X, I \rangle$ is any betweenness relation that satisfies totality, then there are two types of triangle involving three distinct vertices: those where each vertex lies between the other two; and those where exactly one of the vertices does not lie between the other two. As in the ultrametric case, we designate the triangle as *equilateral* in the first situation, and as *tall isosceles* in the second. If $\neg I(a, c, b)$ holds, we call $\{a, b\}$ the *short* edge of the triangle, with the other two edges being referred to as *long* (even though there is no quantified notion of length).

We can characterize ultrametrics—among all metrics—in terms of their $\sqcup$ -inducing betweenness behavior.

2.14. Proposition. *A $\sqcup$ -induced betweenness structure satisfies totality if and only if its inducing metric is an ultrametric.*

Proof. If the inducing metric is an ultrametric, totality follows from Theorems 2.10 and 2.12. Now suppose $\langle X, \varrho \rangle$ is a metric space whose $\sqcup$ -induced betweenness structure satisfies totality, and assume ϱ is not an ultrametric. Then—as in the proof of Proposition 2.8—we have $a, b, c \in X$ with $\varrho(a, b) > \varrho(a, c) \sqcup \varrho(c, b)$. Consequently we have $c \notin I^\sqcup(a, b)$; and, by totality, it follows that $b \in I^\sqcup(a, c)$; i.e., $\varrho(a, c) = \varrho(a, b) \sqcup \varrho(b, c) > \varrho(a, c)$, a contradiction. Hence ϱ is an ultrametric after all. $\square$

If a betweenness structure has at most two points, then that betweenness relation is unique; and trivially satisfies both totality and antisymmetry. The following tells us that this is the only consistency between these two properties.

2.15. Proposition. *If a betweenness structure satisfies both totality and antisymmetry, then it must have at most two points.*

Proof. Suppose $\langle X, I \rangle$ is a betweenness structure that satisfies both totality and antisymmetry, and has three distinct points $a, b, c \in X$. Totality implies that one of these points lies between the other two; say $I(a, c, b)$ holds. Then, by antisymmetry, $I(a, b, c)$ does not hold; and by symmetry, neither does $I(c, b, a)$. A second application of totality gives us $I(c, a, b)$. But then—symmetry again—we conclude that both $I(b, c, a)$ and $I(b, a, c)$ hold, contradicting antisymmetry. $\square$

3. BETWEENNESS $+$ -INDUCED BY METRICS OVER A FINITE VALUE SET

Continuing the thread of Example 2.3 (iii), where metric spaces are viewed as complete graphs with edges labeled from a set Λ of positive real numbers, define a betweenness structure $\langle X, I \rangle$ to be Λ^+ -**induced** (resp., $\Lambda^\sqcup$ -**induced**) if I is $+$ -induced (resp., $\sqcup$ -induced) by a $(\Lambda \cup \{0\})$ -valued metric (resp., ultrametric) on X . In the present section we focus on $+$ -inducement; we treat $\sqcup$ -inducement in the next.

Denoting by $\mathcal{B}(\Lambda^+)$ the class of betweenness structures that are Λ^+ -induced, we have the following result that continues the discussion in Remarks 1.4 and 2.13 (ii).

3.1. Theorem. *If the label set Λ is finite, then $\mathcal{B}(\Lambda^+)$ is an elementary class, axiomatizable by a set of universal sentences.*

Proof. It is easy to check that $\mathcal{B}(\Lambda^+)$ is closed under substructures, regardless of whether Λ is finite. If Λ is actually finite, then it is also easy to check that $\mathcal{B}(\Lambda^+)$ is closed under the taking of ultraproducts. Using classical model-theoretic techniques; particularly [10, Theorems 4.1.12 and 5.2.4, and Corollary 4.3.13], we conclude that $\mathcal{B}(\Lambda^+)$ is universally axiomatizable. $\square$

3.2. Remark. See Section 7, especially Remarks 7.9 (i,ii), for more on the axiomatizability of metric betweenness.

In this section we make the simplifying assumption that our label sets Λ are such that each complete graph labeled with elements of Λ specifies an honest metric. This happens if and only if each member of Λ is no greater than twice any of the others, and we call such a label set **metrical**. The metrical label sets are exactly those that are contained in some real interval $[r, 2r]$, where $r > 0$; and, for the purposes of analyzing betweenness, we lose no generality in assuming $r = 1$ in the sequel.

We first show that metrical label sets behave as if they were finite when it comes to how they induce betweenness. Recall that a betweenness structure is *trivial* if all its nondegenerate intervals are gaps; i.e., if it satisfies the following universal statement.

- (*Triviality*) $I(u, s, v) \rightarrow (s = u \vee s = v)$.

In terms of complete labeled graphs and $+$ -induced betweenness, triviality amounts to saying that all triangles are $+$ -trivial.

The following assertion shows that infinite metrical label sets induce the same betweenness relations as finite ones do.

3.3. Proposition. *Let $\Lambda \subseteq [1, 2]$ be a metrical label set.*

- (i) *If Λ does not contain $\{1, 2\}$, then $\mathcal{B}(\Lambda^+) = \mathcal{B}(\{1\}^+)$.*
- (ii) *If Λ properly contains $\{1, 2\}$, then $\mathcal{B}(\Lambda^+) = \mathcal{B}(\{1, \frac{3}{2}, 2\}^+)$.*

Proof. The first assertion is obvious. As for the second, notice that in any Λ -edge-labeled complete graph, one may replace each label $\lambda \in \Lambda \setminus \{1, 2\}$ with $\frac{3}{2}$ and not alter the $+$ -induced betweenness relation. $\square$

Clearly in the case (i) above, $\mathcal{B}(\Lambda^+) = \mathcal{B}(\{1\}^+)$ consists of all models of the triviality axiom. Case (ii) is a bit more interesting, and we consider it in Theorem 3.5 below. The most interesting case of all, where $\Lambda = \{1, 2\}$, will occupy most of this section.

Any (simple) graph may be viewed as a $\{0, 1, 2\}$ -valued metric space: just add the necessary edges to make the graph complete; then assign label 1 to the original edges and 2 to the new ones. This process is easily reversed: take a $\{1, 2\}$ -edge-labeled complete graph representing the given metric space, delete the edges labeled 2, and then remove the remaining labels.

A betweenness structure is **M-induced** if it is Λ^+ -induced for some metrical label set Λ ; i.e., if it is $\{1, \frac{3}{2}, 2\}^+$ -induced. In view of the last paragraph, the structure is termed **graph-induced** if it is $\{1, 2\}^+$ -induced. In an M-induced betweenness structure, $I(a, c, b)$ holds for distinct a, b, c precisely when the three points are the vertices of a short isosceles triangle which is *integral*—i.e., no edges with label $\frac{3}{2}$ —and which has c as apex.

3.4. Remark. There is another well-known way for a *connected* (simple) graph to induce a metric, and therefore a notion of betweenness: define $\varrho(a, b)$, for distinct vertices a, b , to be the shortest length of an edge path joining them, and set $I = I_\varrho^+$. The two metrics (when defined for a connected graph) are quite different, but they do agree in the special case where the graph has the property that for each two distinct vertices there is a third vertex adjacent to each of the two. In particular, this happens in the case of *Rado graphs*.⁷

Owing to how $+$ -induced betweenness works in a $(\Lambda \cup \{0\})$ -valued metric space when Λ is metrical, it is clear that every M-induced betweenness structure must satisfy the following universal first-order sentence.

- (*Strong Transitivity*) $(I(u, s, v) \wedge I(u, t, s) \wedge s \neq v) \rightarrow (t = u \vee t = s)$,

which says that if a, b, c are distinct points and $I(a, c, b)$ holds, then $I(a, c)$ and $I(c, b)$ are gaps.

In the presence of the three basic betweenness axioms above, strong transitivity formally implies both transitivity and reversal (hence also antisymmetry). It does not imply convexity, however, as is witnessed by the graph consisting of four vertices making up a square circuit, and a fifth vertex adjacent only to two diagonally opposite vertices of the square.

It is natural to conjecture—in the footsteps of Theorems 1.3, 2.11, and 2.12—that a betweenness structure is M-induced if and only if it satisfies strong transitivity.

⁷These are fascinating countable graphs—also known as *random graphs*—studied by Richard Rado, Paul Erdős, and Alfréd Rényi, *inter alia*. They are characterized by the condition that for any two disjoint finite—possibly empty—sets of vertices there is a new vertex that is adjacent to every vertex in one of the sets and not adjacent to any vertex in the other. All Rado graphs are isomorphic to one other, and are *ubiquitous* in the sense of Remark 8.8 below.

This is not true however (see Example 3.9 below): another universally-expressible condition, holding for all M -induced betweenness structures, is essential. For the sake of intelligibility, we state it informally, and to do this we first define a non-degenerate doubleton set $\{a, b\}$ to be **extendable** if there is some $c \notin \{a, b\}$ with either $I(c, a, b)$ or $I(a, b, c)$ holding. Then we have:

- (*Extendability*) If a, b, c are distinct points where $\{a, c\}$ and $\{b, c\}$ are extendable and $\{a, b\}$ is a non-gap, then $I(a, c, b)$ holds.

3.5. Theorem. *A betweenness structure is M -induced if and only if it satisfies both strong transitivity and extendability.*

Proof. If $\langle X, \varrho \rangle$ is a $\{0, 1, \frac{3}{2}, 2\}$ -valued metric space, then strong transitivity is clear. As for extendability, suppose $a, b, c \in X$ are distinct, where $\{a, c\}$ and $\{b, c\}$ are extendable and $\{a, b\}$ is not a gap. Then it follows by definition that $\varrho(a, c) = \varrho(c, b) = 1$ and $\varrho(a, b) = 2$. Hence $I^+(a, c, b)$ holds.

Now let $\langle X, I \rangle$ be a betweenness structure satisfying strong transitivity and extendability. We wish to find a $\{0, 1, \frac{3}{2}, 2\}$ -valued metric ϱ on X such that $I = I^+_\varrho$. As a candidate, define $\varrho(a, b)$, for a, b distinct, to be 1 if $\{a, b\}$ is extendable, to be 2 if $\{a, b\}$ is a non-gap, and to be $\frac{3}{2}$ otherwise. First note that if $\{a, b\}$ is extendable, then it is a gap, due to strong transitivity; hence ϱ is well defined.

To show $I \subseteq I^+$, suppose $I(a, c, b)$ holds for fixed $a, c, b \in X$. Because I satisfies the basic axioms, we may assume without loss of generality that the three points are distinct. Then both $\{a, c\}$ and $\{b, c\}$ are extendable and $\{a, b\}$ is a non-gap, so we know that $\varrho(a, c) = \varrho(c, b) = 1$ and $\varrho(a, b) = 2$. Therefore $I^+(a, c, b)$ holds.

We next show that $I^+ \subseteq I$. If $I^+(a, c, b)$ holds for fixed $a, b, c \in X$, we use the fact that I^+ satisfies the basic axioms to be able again to assume a, b, c are distinct. Then we have $\varrho(a, c) = \varrho(c, b) = 1$ and $\varrho(a, b) = 2$, telling us that $\{a, c\}$ and $\{c, b\}$ are extendable, and that $\{a, b\}$ is not a gap. By the extendability axiom, we infer that $I(a, c, b)$ holds, and the proof is complete. $\square$

Note that strong transitivity explicitly involves four variables, all universally quantified; hence checking failure of this axiom in a betweenness structure involves looking at all subsets with at most four points. Extendability, as stated informally above, nominally involves three variables, but in the formal expression of the axiom, there must be extra variables to account for two pairs being extendable and one more pair being non-gap. This involves six variables in all; hence we have the following consequence of Theorem 3.5.

3.6. Corollary. *If a betweenness structure fails to be M -induced then it has a substructure, with at most six points, which fails to be M -induced.*

We will see later in this section (Corollary 3.18) that if a betweenness structure fails to be graph-induced then it has a substructure with at most five points, which fails to be graph-induced. (This is best possible, since Proposition 3.11 below tells us that all four-point M -induced structures are graph-induced.)

We now address the question of whether satisfying strong transitivity implies being M -induced. First define a betweenness structure I on X to be **subinduced** by a particular antecedent structure on X if $I \subseteq J$ for some betweenness relation J that is induced by that antecedent structure. So terms like *road-subinduced*, *+subinduced*, etc., all make good sense; in particular we are interested in Λ^+ -subinducement in this section.

3.7. Theorem. *A betweenness structure is M-subinduced if and only if it is graph-subinduced, if and only if it satisfies strong transitivity.*

Proof. Being graph-subinduced trivially implies being M-subinduced.

Assume $\langle X, I \rangle$ is M-subinduced; i.e., there is a $\{0, 1, \frac{3}{2}, 2\}$ -valued metric ϱ on X such that $\varrho(a, c) + \varrho(c, b) = \varrho(a, b)$ whenever $I(a, c, b)$ holds. Then it is clearly impossible for there to be distinct a, b, c, d with $I(a, c, b)$ and $I(a, d, c)$ both holding; hence strong transitivity is satisfied.

Suppose $\langle X, I \rangle$ satisfies strong transitivity; let $x, y \in X$ be distinct. Define $\varrho(x, y)$ to be 1 if $\{x, y\}$ is extendable, and to be 2 otherwise. If $I(a, c, b)$ holds for distinct $a, b, c \in X$, then $\{a, c\}$ and $\{c, b\}$ are trivially extendable; hence $\varrho(a, c) = \varrho(c, b) = 1$, by definition. By strong transitivity, every extendable pair of points is a gap. Since $\{a, b\}$ is demonstrably not a gap, we conclude that $\varrho(a, b) = 2$; hence $I^+(a, c, b)$ holds, and $\langle X, I \rangle$ is therefore graph-subinduced. $\square$

From the discussion just prior to Corollary 3.6, we now have the following consequence of Theorem 3.7.

3.8. Corollary. *If a betweenness structure fails to be graph-subinduced then it has a substructure, with at most four points, which fails to be graph-subinduced.*

3.9. Example. We present a graph-subinduced betweenness structure that does not satisfy extendability; hence being graph-subinduced does not imply being M-induced.

We let $X = \{a, b, c, d, e, f\}$ be a six-point set in the plane, arranged as the vertices $\{a, b, c\}$ of an equilateral triangle, with d, e, f the midpoints of the line segments determined by $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$, respectively. Fix a $\{0, 1, 2\}$ -valued metric ϱ such that $\varrho(a, f) = \varrho(d, f) = \varrho(b, d) = \varrho(d, e) = \varrho(c, e) = \varrho(c, f) = 1$ and $\varrho(a, d) = \varrho(b, e) = \varrho(e, f) = 2$. Let $I = I^+ \setminus \{\langle e, d, f \rangle\}$. Then $\langle X, I \rangle$ is graph-subinduced, where $I(e, d, f)$ does not hold. However, both $\{d, e\}$ and $\{d, f\}$ are extendable and $\{e, f\}$ is a non-gap, all relative to I . Hence $\langle X, I \rangle$ fails to satisfy extendability. By Theorem 3.5, $\langle X, I \rangle$ is not M-induced.

By Theorem 3.1, we know that the class of graph-induced betweenness structures is universally axiomatizable. However, strong transitivity and extendability are insufficient for such an axiomatization.

3.10. Example. We present an M-induced betweenness structure that is not graph-induced; hence satisfying both strong transitivity and extendability is not sufficient for a betweenness structure to be graph-induced.

We let $X = \{a, b, c, d, e\}$ be a five-point set arrayed as the vertices of a regular pentagon in the plane. All of the five outer edges are labeled 1, one of the five inner edges is labeled $\frac{3}{2}$, and the remaining edges are labeled 2. Let I be the resulting $+$ -induced betweenness relation, with $\{c, e\}$ the edge with label $\frac{3}{2}$. The question is whether there is a $\{0, 1, 2\}$ -valued metric ϱ that $+$ -induces I , and the answer is no: first note that all the doubletons labeled 1 are extendable and all those labeled 2 are non-gaps; hence ϱ would have to agree with the original labeling when the labels come from $\{1, 2\}$. Note also that the edge defined by $\{c, e\}$ is a common edge of three triangles; two of them with remaining labels 1 and 2, and one with remaining labels

both 1. Thus relabeling $\{c, e\}$ with either 1 or 2 forces a proper enlargement of I ; so $I \neq I_\varrho^+$.

Example 3.10 is *best possible*, in the following sense.

3.11. Proposition. *Every M -induced betweenness structure with at most four points is graph-induced.*

Proof. The assertion is easily verified for at most three points, so we lose no generality in assuming that our betweenness structure has exactly four points. Hence let $\langle X, I \rangle$ be $\{1, \frac{3}{2}, 2\}^+$ -induced, where $X = \{a, b, c, d\}$. If all of the triangles of X are $+$ -trivial, then we may relabel all the edges with 1 (or with 2) and obtain a $\{1, 2\}$ -labeling that induces I .

We may assume that at least one edge in X is labeled $\frac{3}{2}$; otherwise there is nothing to prove. With this assumption, there are at most two triangles of X that have a chance of being $+$ -nontrivial. If there is just one, say it is $\{a, b, c\}$, then we may relabel the three edges with end vertex d with 2, and retain the original betweenness structure.

Now assume that exactly two triangles are $+$ -nontrivial, say they are $\{a, b, c\}$ and $\{b, c, d\}$. If the common edge $\{b, c\}$ is labeled 2, then all edges other than $\{a, d\}$ are labeled 1. Hence relabeling edge $\{a, d\}$ with 1 will retain the original betweenness structure. If $\{b, c\}$ is labeled 1, there are two subcases to consider: (1) the two edges labeled 1 (or 2) share a vertex, in which case we relabel $\{a, d\}$ with 1; and (2) those two edges do not share a vertex, in which case we relabel $\{a, d\}$ with 2. Either way, we retain the original betweenness structure. $\square$

Example 3.10 serves as inspiration for stating the following universally axiomatizable condition, one which holds for all graph-induced (but not all M -induced) betweenness structures.

- (*Dichotomy*) If a, b, c, d are distinct points, $\{a, c\}$, $\{b, c\}$, and $\{b, d\}$ are extendable, and $\{a, d\}$ is a non-gap, then $c \in I(a, b)$ or $b \in I(a, d)$.

Clearly the five-point example 3.10 fails to satisfy the dichotomy axiom; and for a while, we tried to prove that a betweenness structure is graph-induced if and only if it satisfies strong transitivity, extendability, and dichotomy. The *only if* half of the assertion is easy to show, but the *if* half is false, as the following example shows.

3.12. Example. We present an M -induced betweenness structure that satisfies dichotomy, but is not graph-induced; hence satisfying strong transitivity, extendability, and dichotomy is not sufficient for a betweenness structure to be graph-induced.

We let $X = \{a, b, c, d, e, f, g, h, i, j\}$ be a ten-point set arrayed as the vertices of a pentagram—i.e., a regular five-pointed star—in the plane. We label one of the points of the star with a , then proceed clockwise with the labeling of the remaining vertices. Only the outermost triangles are $+$ -nontrivial, and we stipulate $I(a, b, j)$, $I(b, c, d)$, $I(d, f, e)$, $I(f, h, g)$, and $I(h, j, i)$. In any $\{1, \frac{3}{2}, 2\}$ -labeling of X that induces I , all the edges of the outer triangles are either extendable or non-gap; hence they will be labeled from $\{1, 2\}$ appropriately. And if we label the remaining 30 edges with $\frac{3}{2}$, we obtain a $\{0, 1, \frac{3}{2}, 2\}$ -valued metric on X , one that $+$ -induces I . Hence, by Theorem 3.5, $\langle X, I \rangle$ satisfies strong transitivity and extendability. Note that there are no four points x, y, z, w of X where $\varrho(x, z) = \varrho(z, y) =$

$\varrho(y, w) = 1$ and $\varrho(w, x) = 2$. Hence dichotomy also holds for $\langle X, I \rangle$, as its premise is never satisfied.

Finally, suppose ϱ is a $\{0, 1, 2\}$ -valued metric that $+$ -induces I . Since $\varrho(f, h) = \varrho(h, j) = 1$ and $\neg I(f, h, j)$, we must have $\varrho(f, j) = 1$. But $\varrho(b, j) = 1$ and $\neg I(b, j, f)$ also; hence $\varrho(b, f) = 1$. Since $\varrho(d, f) = 1$ and $\varrho(b, d) = 2$, we have forced $I(b, f, d)$, a contradiction. This shows that $\langle X, I \rangle$ is not graph-induced.

Example 3.12 suggests the possibility that the class of graph-induced betweenness structures may have no simple axiomatization such as that afforded by Theorem 3.5; in fact it may not be finitely axiomatizable at all. We now set about showing that this class does indeed have a finite universal axiomatization, which is relatively simple. However, the only proof we have been able to find takes a circuitous route using additional axioms that are somewhat less straightforward than the ones so far considered.

For the remainder of this section we use the following informal notational conventions: (1) distinct lower-case letters are presumed to denote distinct points; (2) the expressions acb , ab , and $a.b$ respectively abbreviate the statements $I(a, c, b)$, $\exists x (I(x, a, b) \vee I(a, b, x))$ (i.e., $\{a, b\}$ is extendable), and $\exists x I(a, x, b)$ (i.e., $\{a, b\}$ is a non-gap); and (3) $ab = t$ denotes the metric assignment $\varrho(a, b) = t$, where $t = 0, 1$.

So in this notation, strong transitivity and extendability become $a.b \rightarrow \neg ab$ and $(ac \wedge bc \wedge a.b) \rightarrow acb$, respectively. In light of Theorem 3.5, it now suffices to characterize the graph-induced betweenness structures among the M-induced ones; in particular strong transitivity and extendability are tacitly part of all further axiomatizations in this section.

Consider the following three new conditions.

- (*Parity*) $(bc \wedge cd \wedge \neg bd \wedge \neg bcd) \rightarrow ((ab \wedge ad) \vee (\neg ab \wedge \neg ad))$
- (*Lifting*) $(cad \wedge ab) \rightarrow (bac \vee bad \vee cbd)$
- (*Co-lifting*) $(bdc \wedge ab) \rightarrow (abd \vee adc \vee bac)$.

We first note that parity alone is not syntactically transformable—via the well-known prenex normal form process—into a universal sentence. However, when we combine it with lifting, we are at least able to obtain a *first-order* axiomatization for the graph-induced betweenness structures. As a second step, we incorporate co-lifting in order to replace parity+lifting with a simple axiom that is universal.

3.13. Theorem. *An M-induced betweenness structure is graph-induced if and only if it satisfies parity and lifting.*

Proof. Start with a graph-induced betweenness structure $\langle X, I \rangle$. To see that parity holds, suppose we have four distinct points $a, b, c, d \in X$. By the premise $bc \wedge cd$ we have $bc = cd = 1$, and from $\neg bcd$ we also have $bd = 1$. Suppose the conclusion is false; without loss of generality, assume $\neg ab \wedge ad$ now holds. Since ad holds, there is a point $e \notin \{a, d\}$ with either ade or dae holding. Suppose it is ade . Then $ad = de = 1$ and $ae = 2$. Since $\neg bd$ holds, it follows that $\neg bde$ also holds; hence $be = 1$. Since $\neg ab$, we have $ab = 2$. But then adb holds; hence so does bd , a contradiction. If dae holds, we similarly derive the same contradiction.

As for lifting, again let $a, b, c, d \in X$ be distinct. From the premise of the axiom, we have $ab = ac = ad = 1$ and $cd = 2$. If $\neg bac$, then we have $bc = 1$. If also $\neg bad$, then $bd = 1$. But then we have cbd .

Now suppose $\langle X, I \rangle$ is a betweenness structure satisfying strong transitivity, extendability, parity, and lifting; we wish to construct a $\{0, 1, 2\}$ -valued metric on X that $+$ -induces I . This labeling needs to be done in such a way that: (1) whenever acb holds, we have $ac = bc = 1$ and $ab = 2$; and (2) whenever $\{a, b, c\}$ is a $+$ -trivial I -triangle, it cannot be the case that exactly two edges are labeled 1. Our proposed labeling is only slightly more complicated than the one offered in the proof of Theorem 3.7: for $x, y \in X$ distinct, define $xy = 1$ if and only if either xy , or $xz \wedge yz \wedge \neg xzy$ holds for some $z \in X$.

Suppose acb holds; we wish first to show that $I^+(a, c, b)$ is also true. Immediately we have $ac \wedge bc \wedge a.b$; hence our labeling definition gives $ac = bc = 1$. By strong transitivity, $\neg ab$. So if it were the case that $ab = 1$, our labeling definition would require the existence of some $d \in X$ with $ad \wedge bd \wedge \neg adb$ holding. But $a.b$ is true, and this contradicts extendability. Hence $ab = 2$, and therefore $I^+(a, c, b)$ holds. What remains to show is the converse.

Suppose $b, c, d \in X$ are such that $bd = bc = 1$ and $cd = 2$. We need to show that cbd holds. Indeed, suppose otherwise; that $bd = bc = 1, cd = 2$, and $\neg cbd$. If bd and bc were both true, then we would have assigned $cd = 1$, by the labeling rule above. Therefore either $\neg bd$ or $\neg bc$ is true; say, without loss of generality, it is $\neg bd$. Since $cd = 2$, we immediately know $\neg cd$. Since $bd = 1$ and $\neg bd$ holds, there must be some $a \in X$ with $ab \wedge ad \wedge \neg bad$ (again by our labeling rule). Since we now have $ab \wedge ad \wedge \neg bd \wedge \neg bad \wedge \neg cd$, parity implies that $\neg bc$ holds. But we set $bc = 1$. Therefore there is some $e \in X$ with $be \wedge ce \wedge \neg bec$ holding. Since $be \wedge ce \wedge \neg bc \wedge \neg bec \wedge ab$ now holds, a second application of parity gives us ac . Since $ac \wedge ad$ and $cd = 2 \neq 1$, we know cad must hold; hence $c.d$ holds. By the lifting axiom, since $cad \wedge ab$, we get $bac \vee cbd \vee bad$. But $\neg cbd \wedge \neg bad$, so bac holds. But then $b.c$, and thus bec holds by the extendability axiom. This is a contradiction, and the proof is complete. $\square$

3.14. Remark. The dichotomy axiom is never used in the proof of Theorem 3.13; hence it follows from the conjunction of strong transitivity, extendability, parity, and lifting.

In order to obtain the desired characterization of graph-induced betweenness structures in terms of finitely many axioms that are *universal*, we present the following condition, which is easily seen to hold in all graph-induced structures. It is based on the geometric idea that in such a structure, if we are given four distinct points such that one of the four associated triangles is short isosceles (i.e., $+$ -nontrivial) and one of the three edges not on that triangle is extendable, then one of the three other triangles is also short isosceles. We will then be able to replace parity and lifting in Theorem 3.13 with the following universal condition.

$$\bullet \text{ (Graphing) } (abc \wedge (ad \vee bd \vee cd)) \rightarrow (adc \vee abd \vee bad \vee bcd \vee cbd).$$

Key to our characterization is the following.

- 3.15. Lemma.** (i) *Any betweenness structure satisfying lifting and co-lifting also satisfies graphing.*
(ii) *Any graph-induced betweenness structure satisfies lifting, co-lifting, and graphing.*
(iii) *Any M -induced betweenness structure satisfies graphing if and only if it satisfies both lifting and co-lifting.*

- (iv) Any M -induced betweenness structure satisfying parity also satisfies co-lifting.
- (v) In any M -induced betweenness structure satisfying lifting, parity and co-lifting are equivalent.

Proof. Ad (i). Let $\langle X, I \rangle$ be a betweenness structure, with $a, b, c, d \in X$ distinct, abc holding, and one of the three remaining edges being extendable. If either ad or cd holds, apply co-lifting; if bd holds, apply lifting.

Ad (ii). Every graph-induced betweenness structure satisfies lifting, by Theorem 3.13. As for co-lifting, suppose $a, b, c, d \in X$ are distinct, with $bdc \wedge ab$ holding. Then we have $ab = bd = cd = 1$ and $bc = 2$. If we also have $\neg abd \wedge \neg adc$, then $ad = 1$ and $ac = 1$. But this tells us that bac holds. Graphing now follows from (i).

Ad (iii). Let $\langle X, I \rangle$ be an M -structure. Because of (i), it remains to show that satisfying graphing implies satisfying both lifting and co-lifting. Given $a, b, c, d \in X$ distinct, with $abc \wedge bd$ holding, and using strong transitivity and extendability, we see that the only possible instances of betweenness involving d are abd , cbd , and adc . Graphing thus ensures one of these; hence lifting holds. We prove co-lifting from graphing similarly.

Ad (iv). Suppose co-lifting fails with abc and ad holding but dab, dbc , and adc all failing. By extendability, $\neg adc$ implies $\neg cd$. By strong transitivity, abc implies $\neg ac$. If $\neg bd$, then since $\neg bad$, parity implies $bc \Leftrightarrow cd$, which is false. If bd , then since $\neg cbd$, parity implies $ac \Leftrightarrow ad$, which is also false.

Ad (v). One direction of the equivalence was established in (iv). For the converse, suppose extendability, strong transitivity, lifting, and co-lifting all hold and that parity fails. We indicate this assumption with the following list of six conditions:

$$(\alpha) \quad bc, cd, ab, \neg bd, \neg ad, \neg bcd,$$

and devote the rest of the argument to showing $\neg ab$, contradicting (α) . Note that, from (α) , we have the additional list

$$(\mu) \quad \neg bac, \neg abd, \neg bad.$$

By co-lifting, acd and ab would imply $bac \vee bcd \vee abd$, which fails by (α) . Therefore we have

$$(\zeta) \quad \neg acd.$$

Suppose $x \notin \{a, b, d\}$.

If abx and dx both hold, then by co-lifting, we infer $bx d \vee abd \vee adx$, which is false by (α) . Hence we conclude

$$(\xi) \quad abx \Rightarrow \neg dx \text{ (so, letting } x = c, \text{ we have } \neg abc).$$

If dcx and ax both hold, then by co-lifting, we infer $axc \vee acd \vee dax$. But dax and acd fail by (α) and (ζ) , so axc holds, ac is a non-gap, and abc holds, contradicting (ξ) . Thus

$$(\tau) \quad dcx \Rightarrow \neg ax.$$

If dcx holds, then by lifting with bc , we have $bcd \vee bcx \vee dbx$. But $bcd \vee dbx$ is false by (α) , and so $dcx \Rightarrow bcx$. But if bcx holds, then by lifting with cd , we have $bcd \vee dcx \vee bdx$. But $bcd \vee bdx$ is false by (α) , and so $bcx \Rightarrow dcx$. If cdx holds, then by co-lifting with bc , we have $bcd \vee bdx \vee cbx$. But $bcd \vee bdx$ is false by (α) , and so

$cdx \Rightarrow cbx$. And if cbx holds, then by co-lifting with cd , we have $bcd \vee dbx \vee cdx$. But $bcd \vee dbx$ is false by (α) , and so $cbx \Rightarrow cdx$. So we now have

$$(\omega) \quad bcx \Leftrightarrow dcx \text{ and } cbx \Leftrightarrow cdx.$$

Then: (1) (α) implies $\neg adx$ and cd , and (ζ) implies $\neg acd$. So by lifting, $acx \wedge cd \Rightarrow dcx$, which, by (ω) , implies bcx and thus $\neg bx$; and (2) (α) implies $\neg dax$ and cd , and (ζ) implies $\neg acd$. So by co-lifting, $cax \wedge cd \Rightarrow cdx$, which by (ω) implies cbx and thus bx . Hence

$$(\gamma) \quad acx \Rightarrow \neg bx \text{ and } cax \Rightarrow bx.$$

Then: (3) by (ξ) , abx implies $\neg dx$, which implies $\neg cdx$, and by (ω) , $\neg cbx$. Also $\neg abc$ by (ξ) . By lifting, $abx \wedge bc \Rightarrow acx$, and then by (γ) , $\neg bx$ and $\neg abx$; and (4) by (ω) , bcx implies dcx and thus by (τ) , $\neg ax$ and so $\neg bax$. Also $\neg abc$ by (ξ) . By co-lifting, $bax \wedge bc \Rightarrow cax$, and then by (γ) , bx and $\neg bax$. Hence

$$(\delta) \quad \neg abx \text{ and } \neg bax.$$

(μ) and (ξ) also give $\neg abc$, $\neg abd$, $\neg bac$, and $\neg bad$. So by (δ) , we get $\neg ab$, and our contradiction. $\square$

The desired universal axiomatization is now an easy consequence of the above.

3.16. Theorem. *For an M -induced betweenness structure, the following conditions are equivalent:*

- (1) *Being graph-induced.*
- (2) *Satisfying the graphing axiom.*
- (3) *Satisfying the conjunction of the lifting and co-lifting axioms.*

Proof. Any graph-induced betweenness structure satisfies graphing, by Lemma 3.15 (ii). Hence we have $(1) \Rightarrow (2)$.

Suppose our M -induced structure satisfies graphing. Then, by Lemma 3.15 (iii) it satisfies lifting and co-lifting; and hence, by Lemma 3.15 (v), it also satisfies lifting and parity. Hence it is graph-induced, by Theorem 3.13. This gives us $(2) \Rightarrow (1)$. The equivalence of (2) and (3) is the content of Lemma 3.15 (iii). $\square$

When we syntactically analyze the axioms graphing, lifting, and co-lifting, we see that a failure on the part of a given betweenness structure to satisfy any of these axioms may be detected through the investigation of substructures with at most five points. On the other hand, failure to satisfy extendability by itself apparently requires investigating six-point substructures. Despite this, it is possible to obtain a graph-induced analogue of Corollary 3.6 that involves only five-point substructures. (Because of Proposition 3.11, this cannot be further improved to four-point substructures.)

3.17. Lemma. *If a betweenness structure satisfies strong transitivity and graphing, but not extendability, then it has a substructure with at most five points, that fails extendability.*

Proof. Suppose extendability fails. This means we have distinct points a, b, c , with ab , ac , and $b.c$ holding, but bac failing. Since $\{b, c\}$ is a non-gap, there is a witness; i.e., some $d \notin \{b, c\}$ so that bdx holds. Since $\neg bac$, we also know $d \neq a$. By graphing, each of the two premises $bdx \wedge ab$ and $bdx \wedge ac$ implies the disjunction $bac \vee adb \vee abd \vee acd \vee adc$. The first disjunct fails by assumption, and the second

and fifth fail because of strong transitivity. This leaves $abd \vee acd$, saying that the fourth point d witnesses both $b.c$ and at least one of ab , ac . While a fifth point e may be required to witness the remaining condition, that will suffice; we do not need a sixth. Hence extendability fails for the (at most) five-point substructure $\{a, b, c, d, e\}$. $\square$

The advertised analogue of Corollary 3.6 is now easy.

3.18. Corollary. *If a betweenness structure fails to be graph-induced then it has a substructure, with at most five points, which fails to be graph-induced.*

Proof. Suppose a given betweenness structure fails to be graph-induced. By Theorem 3.16, it fails at least one of strong transitivity, extendability, and graphing. If strong transitivity fails, there is a four-point substructure in which that axiom fails. If graphing fails, there is a five-point witnessing substructure. (We may replace graphing here with the conjunction of lifting and co-lifting.) If extendability is the only axiom that fails, then Lemma 3.17 tells us there is also a five-point witnessing substructure. $\square$

3.19. Remark. If we replace *graphing* with *co-lifting* in the hypothesis of Lemma 3.17, then the current argument can be easily adjusted to show that the point d witnesses all three assumptions ab , ac , and $b.c$. Hence a *four*-point substructure witnesses the failure of extendability. This fact does not readily help us in proving Corollary 3.18, however, as it is unclear whether graphing implies co-lifting in the absence of extendability.

4. BETWEENNESS $\sqcup$ -INDUCED BY ULTRAMETRICS OVER A FINITE VALUE SET

In this section metric spaces continue to be viewed as complete graphs with edges labeled from a finite set Λ of positive real numbers. When the intended metric is an ultrametric, however, all triangles in such labeled graphs are either tall isosceles or equilateral (see Section 2).

A betweenness structure $\langle X, I \rangle$ is $\Lambda^\sqcup$ -**induced** if it is $\sqcup$ -induced by a $(\Lambda \cup \{0\})$ -valued ultrametric on X . Let $n \geq 1$ be a fixed positive integer. Because only the total order structure of an n -element label set Λ is relevant—not the additive—we lose no generality in fixing Λ to be $\Lambda_n := \{1, 2, \dots, n\}$. Denote by $\mathcal{B}(\Lambda_n^\sqcup)$ the class of all $\Lambda_n^\sqcup$ -induced betweenness structures. The aim of this section is a first-order characterization of $\mathcal{B}(\Lambda_n^\sqcup)$ in the spirit of Section 3.

Note that with $\Lambda_1^\sqcup$ -induced betweenness structures, all triangles are equilateral and each point lies between any two others. Hence $\mathcal{B}(\Lambda_1^\sqcup)$ is easy to characterize with the single axiom $u \neq v \rightarrow I(u, w, v)$. The cases $n \geq 2$ are more interesting; first we comment that in the case $n = 2$, we may mimic what we did in the last section and convert an ultrametric space labeled with 1s and 2s into an unlabeled (simple) graph: delete all edges labeled 2, and remove the rest of the labels. For the reverse, where we would normally label each edge of a graph with 1, connect all other vertices with new edges, and label them all with 2, we need to be more careful: if $\{a, c\}$ and $\{c, b\}$ are edges while $\{a, b\}$ is not, this process would label $\{a, b\}$ with 2, creating a short isosceles triangle. For the remedy, we need to restrict ourselves to graphs whose connected components are complete; i.e., *cliques* in the graph. Such graphs are commonly referred to as *cluster* graphs; so we define a betweenness structure to be **cluster graph-induced** precisely when it is $\Lambda_2^\sqcup$ -induced. We will

see that it is noticeably easier to characterize cluster graph-induced betweenness structures than it is to characterize graph-induced ones (as in the last section).

In order to obtain our characterization, it will be useful to talk about *chains*, as we did in Section 1. Given $\langle X, I \rangle$ a betweenness structure and $a \in X$, recall the binary slice $\leq_a$, given by $x \leq_a y$ if $I(a, x, y)$. When $\langle X, I \rangle$ also satisfies transitivity, each $\leq_a$ is a pre-order, with least element a . What keeps it from being a partial order is the absence of antisymmetry, but we may still define a set $C \subseteq X$ to be an *a-chain* (generalizing the definition in the paragraph prior to Proposition 1.2) as follows: for $x, y \in X$, define $x <_a y$ to mean that $x \leq_a y$ but $y \not\leq_a x$. Then $C \subseteq X$ is an *a-chain* if for each two distinct $x, y \in C$ we have either $x <_a y$ or $y <_a x$. As before, a **chain** is a subset that is an *a-chain* for some $a \in X$; a chain's *length* is its cardinality.

When $\langle X, I \rangle$ satisfies totality (along with transitivity), and $b_1 <_a b_2$ for distinct a, b_1, b_2 , we may view the triple $\{a, b_1, b_2\}$ as a tall isosceles triangle T_1 with short edge $\{a, b_1\}$ and apex b_2 . More generally, if $k \geq 1$ and $b_1 <_a b_2 <_a \dots <_a b_k <_a b_{k+1}$ for distinct $a, b_1, \dots, b_{k+1}$, we may view this chain, of length $k + 1$, as a sequence $\langle T_1, T_2, \dots, T_k \rangle$ of k tall isosceles triangles, where, for $1 \leq i \leq k$, T_i is the triangle $\{a, b_i, b_{i+1}\}$, with short edge $\{a, b_i\}$ and apex b_{i+1} . (So the short edge of each successor triangle is a long edge of its predecessor.) We refer to this sequence as a **k-tower over** the edge $\{a, b_1\}$; the integer k is the **height** of the tower. The universal sentence

$$(u \neq v_1 \wedge u \neq v_2 \wedge v_1 \neq v_2) \rightarrow I(u, v_1, v_2)$$

simply says that there is no 1-tower over any edge, and therefore that every point is between any two distinct points—all triangles are equilateral.

By Theorems 2.10 and 2.12, we know that all betweenness structures $\sqcup$ -induced by an ultrametric satisfy transitivity and totality. If the structure is $\Lambda_n^\sqcup$ -induced, then each k -tower involves $k + 1$ distinct values from Λ_n . Hence there can be no n -towers, and this is expressed as the following universal axiom, where we interpret distinct variables as denoting distinct points.

- (*n-Height-restriction*) $\bigvee_{i=1}^n I(u, v_{i+1}, v_i)$.

Note that transitivity and totality add nothing to n -height-restriction when $n = 1$, but for larger values of n they are essential for making sense of viewing chains as towers. We shall see that while adding them to 2-height-restriction does characterize $\mathcal{B}(\Lambda_2^\sqcup)$, the cluster graph-induced betweenness structures, the same ploy does not work in the cases $n \geq 3$. We need to be able to say that if one can build a k -tower over one long edge of a tall isosceles triangle, one can build a k -tower over the other. (As before, distinct variables denote distinct points.)

- (*Tower-symmetry*) $I(u, w, v) \rightarrow (I(t, v, u) \vee I(t, w, v))$.

This axiom—in contrapositive—says that if $\{a, b, c\}$ is tall isosceles with short edge $\{a, b\}$, and if $\{a, c, d\}$ is tall isosceles with short edge $\{a, c\}$, then $\{b, c, d\}$ is tall isosceles with short edge $\{b, c\}$. Along with n -height-restriction, it is intended to force the two long edges of a tall isosceles triangle—as well as all edges of an equilateral triangle—to have the same label in an attempted ultrametric labeling from Λ_n .

4.1. Theorem. *A betweenness structure is $\Lambda_n^\sqcup$ -induced if and only if it satisfies transitivity, totality, n -height-restriction, and tower-symmetry.*

Proof. For each $n \geq 1$ let Γ_n comprise the four axioms, transitivity, totality, n -height-restriction, and tower-symmetry. We first show that if $\langle X, I \rangle$ is $\Lambda_n^\sqcup$ -induced by ultrametric ϱ , then Γ_n holds; and from earlier remarks, this means showing tower-symmetry to be satisfied. Indeed, suppose $a, b, c, d \in X$ are distinct, with $\neg I(a, c, b) \wedge \neg I(a, d, c)$ holding. Then we have $\varrho(a, b) < \varrho(a, c) = \varrho(b, c) < \varrho(a, d) = \varrho(c, d)$. But then, since $\varrho(b, c) < \varrho(c, d)$, triangle $\{b, c, d\}$ is tall isosceles with short edge $\{b, c\}$, so $\neg I(b, d, c)$ holds. This establishes tower-symmetry.

Suppose $\langle X, I \rangle$ satisfies Γ_n . Recall that all triangles of X are either equilateral or tall isosceles. Given $a, b \in X$ distinct, define the **height** $\alpha(a, b)$ of the edge $\{a, b\}$ to be the maximum height of a tower over $\{a, b\}$. Then $0 \leq \alpha(a, b) \leq n - 1$, by n -height-restriction; and we define $\varrho(a, b) = n - \alpha(a, b)$, a function clearly taking its values in Λ_n . We now need to show that ϱ is an ultrametric, and for this it suffices to show that: (1) the two long edges of a tall isosceles triangle have the same height, which is less than that of the short edge; and (2) all three edges of an equilateral triangle have the same height.

Suppose first that $\{a, b, c\}$ is tall isosceles with short edge $\{a, b\}$. Then, by definition of *height*, we have $\alpha(a, b) > \alpha(a, c) \sqcup \alpha(b, c)$, as any k -tower over, say, $\{a, c\}$ becomes a $(k + 1)$ -tower over $\{a, b\}$ by prefixing the triangle $\{a, b, c\}$. We need to show $\alpha(a, c) = \alpha(b, c)$; our proof is by induction on height. If $\alpha(a, b) = 1$, then $\alpha(a, c) = \alpha(b, c) = 0$ and $\varrho(a, c) = \varrho(b, c) = n$.

Fix $1 \leq k \leq n - 2$ and assume that for all tall isosceles triangles $\{x, y, z\}$, with short edge $\{x, y\}$ and $\alpha(x, y) \leq k$, we have $\alpha(x, z) = \alpha(y, z) = \alpha(x, y) - 1$. For our induction step, assume that $\alpha(a, b) = k + 1$. Then we have a $(k + 1)$ -tower $\langle T_0, T_1, \dots, T_k \rangle$ over $\{a, b\}$ and $\langle T_1, \dots, T_k \rangle$ is a k -tower over $\{b, c\}$, where T_1 has third vertex d . By our induction hypothesis, we have $\alpha(b, c) \leq k$, and hence $\alpha(b, d) = \alpha(c, d) = \alpha(b, c) - 1$. Thus we lose no generality in assuming that the tower beyond T_1 is a tower over $\{c, d\}$. If $T'_1 = \{a, c, d\}$, then $\langle T'_1, T_2, \dots, T_k \rangle$ is a k -tower over $\{a, c\}$, showing that $\alpha(b, c) \leq \alpha(a, c)$. Similarly we show the reverse inequality; hence $\alpha(a, c) = \alpha(b, c)$.

Now suppose $\{a, b, c\}$ is equilateral. If all the edge heights are zero, there is nothing to prove; so assume there is a k -tower $\langle T_1, T_2, \dots, T_k \rangle$ over edge $\{a, b\}$, with $T_1 = \{a, b, d\}$. We know that $\neg I(a, d, b)$ holds; suppose it were the case that $I(a, d, c)$ were to hold. Then, since $I(a, c, b)$ holds by assumption, we have $I(a, d, b)$ by transitivity. Thus we infer that all three triangles with vertex d and edges from $\{a, b, c\}$ are tall isosceles with apex d . Hence we may replace our original k -tower with a new one by replacing T_1 with either triangle $\{a, d, c\}$ or $\{b, d, c\}$. Thus all edges of $\{a, b, c\}$ have the same height, and we conclude that ϱ is an ultrametric, as advertised.

That $I = I^\sqcup$ is now almost anticlimactic: If $I(a, b, c)$ holds, one case is that $\neg I(a, c, b)$ also holds. Then, by the proof of the fact that ϱ is an ultrametric, we know that $\varrho(a, b) < \varrho(a, c) = \varrho(b, c)$. Hence $I^\sqcup(a, b, c)$ holds. The other case is that $\{a, b, c\}$ is an equilateral triangle. But then we have shown that in this case all three vertex distances are equal; hence $I^\sqcup(a, b, c)$ holds here as well.

For the reverse inclusion, if $\neg I(a, b, c)$ holds, then $\varrho(a, c) < \varrho(a, b) = \varrho(b, c)$, and hence $\neg I^\sqcup(a, b, c)$ holds. This completes the proof. $\square$

When $n = 2$, tower-symmetry is superfluous, as there are never any towers over a long edge of a tall isosceles triangle. But in the cases $n \geq 3$, it is essential for characterizing $\Lambda_n^\sqcup$ -inducement, as the next example shows.

4.2. Example.

We present a betweenness structure that satisfies transitivity, totality and 3-height-restriction, but which is not in $\mathcal{B}(\Lambda_3^\sqcup)$.

Let $X = \{a, b, c, d\}$, with I described as follows: for triangle $\{a, b, c\}$ we have $I(a, b, c) \wedge I(b, a, c) \wedge \neg I(a, c, b)$; for triangle $\{a, b, d\}$ we have $I(a, b, d) \wedge I(b, a, d) \wedge \neg I(a, d, b)$; for triangle $\{b, c, d\}$ we have $I(b, c, d) \wedge I(c, b, d) \wedge \neg I(b, d, c)$; and for triangle $\{a, c, d\}$ we have $I(a, c, d) \wedge I(c, a, d) \wedge I(a, d, c)$. The first three triangles are tall isosceles, and the fourth is equilateral; so totality is clearly satisfied. It is also clear that 3-height-restriction is satisfied; we have only transitivity to check. For transitivity to fail, we need to force one of the three betweenness assertions $I(a, c, b)$, $I(a, d, b)$, $I(b, d, c)$. If we try to force $I(a, c, b)$, via transitivity, from a premise of the form $I(a, x, b) \wedge I(a, c, x)$, there is no choice for x but to be d . But while $I(a, c, d)$ is true, $I(a, d, b)$ is not. The other two cases are handled similarly, so we are left with the question of being $\Lambda_3^\sqcup$ -induced.

Suppose ϱ is an ultrametric that labels the six edges of $\langle X, I \rangle$ from $\Lambda_3 = \{1, 2, 3\}$. The triangles $\{a, b, c\}$ and $\{b, c, d\}$ constitute a 2-tower over edge $\{a, b\}$; and so we necessarily have $\varrho(a, b) < \varrho(b, c) < \varrho(b, d)$, forcing these values to be 1, 2, 3, respectively. Also, since $\varrho(b, c) = 2$ and $I(b, c, d) \wedge I(c, b, d) \wedge \neg I(b, d, c)$ holds, we have $\varrho(b, d) = \varrho(c, d) = 3$. Since ϱ is an ultrametric, we now have $\varrho(a, c) = \varrho(b, c) = 2$ and $\varrho(a, d) = 3$. But this contradicts the fact that $I(a, d, c)$ holds. Hence $\langle X, I \rangle$ is not $\Lambda_3^\sqcup$ -induced. (So by Theorem 4.1, tower-symmetry must fail.)

In the spirit of Corollaries 3.6, 3.8, and 3.18, we may use Theorem 4.1 to specify when being $\Lambda_n^\sqcup$ -induced fails. Totality involves just three variables (all universally quantified), transitivity and tower-symmetry each involve four, and n -height restriction involves $n + 2 \geq 4$. Hence we have the following.

4.3. Corollary. *Let $n \geq 2$. If a betweenness structure fails to be $\Lambda_n^\sqcup$ -induced then it has a substructure, with at most $n + 2$ points, which fails to be $\Lambda_n^\sqcup$ -induced.*

5. OMITTED PATTERNS

The characterization theorems 3.5, 3.7, 3.16, and 4.1 may be cast in terms not unlike those in Kuratowski's *Graph Theorem* (see, e.g., [14]). This famous result characterizes, in terms of so-called *omitted subgraphs*,⁸ the class of finite graphs that can be (topologically) embedded in the Euclidean plane. In the general spirit of Kuratowski's theorem, we recast our characterization theorems in terms of what we call *omitted patterns*.

Let A be a finite alphabet, with ' $\neg$ ' (abusing notation slightly) a special symbol not in A . By a **particle** over A , we mean a string of the form xyz or $\neg xyz$, where $x, y, z \in A$ are distinct. A **pattern** over A is a finite set of particles. If α is a particle over A , $A|\alpha$ is the set of symbols from A that occur in α . If p is a pattern, $A|p := \bigcup\{A|\alpha : \alpha \in p\}$.

A **realization** of pattern p in a betweenness structure $\langle X, I \rangle$ is an injective function $f : A|p \rightarrow X$ such that: (1) if $xyz \in p$ then $I(f(x), f(y), f(z))$ holds; and (2) if

⁸These are, basically, adjacency-preserving injections into (maybe a subdivision of) the given graph, of a copy of the five-vertex complete graph or of the six-vertex bipartite graph with three vertices in each part.

$\neg xyz \in p$ then $I(f(x), f(y), f(z))$ fails.⁹ A betweenness structure **realizes** (resp., **omits**) a pattern if there exists (resp., does not exist) a realization of the pattern in the structure. Two patterns are **equivalent** if they are realized/omitted by the same betweenness structures; a pattern is **self-contradictory** if for some variables x, y, z it contains both xyz and $\neg xyz$, or any variation of these two particles arising from betweenness symmetry (e.g., zyx and $\neg xyz$).

Let $\mathcal{C}$ be a universal class of betweenness structures (i.e., an elementary class closed under substructures). Then a pattern p is **consistent** with $\mathcal{C}$ if some betweenness structure in $\mathcal{C}$ realizes p . If p is inconsistent with $\mathcal{C}$, we say p is **omitted** by $\mathcal{C}$ (equivalently, $\mathcal{C}$ **omits** p). Immediate from the definitions, all self-contradictory patterns are omitted by all betweenness structures.

A set P of patterns over A is consistent with $\mathcal{C}$ (resp., omitted by $\mathcal{C}$) if the same can be said for each $p \in P$. If P is omitted by $\mathcal{C}$, but also has the property that betweenness structures are in $\mathcal{C}$ whenever they omit each pattern in P , we say that P **characterizes** $\mathcal{C}$. Note that if P characterizes $\mathcal{C}$ and $p \in P$ is self-contradictory, then $P \setminus \{p\}$ characterizes $\mathcal{C}$ as well. Also if $p, q \in P$ are distinct but equivalent, then either may be deleted from P without destroying characterization.

In this section we show how to characterize the classes of betweenness structures featured in the last two sections, in terms of omitted patterns.

Recall the informal notation conventions introduced in Section 3, namely: (1) distinct lower-case letters are presumed to denote distinct points; and (2) the expressions acb , ab , and $a.b$ respectively abbreviate $I(a, c, b)$, $\exists x (I(x, a, b) \vee I(a, b, x))$ (extendability), and $\exists x I(a, x, b)$ (non-gap).

For the purposes of this section, call a first-order formula over the ternary betweenness relation **regular** if it is of the form $\varphi \rightarrow \perp$, where φ is a conjunction of formulas of the form abc and $\neg abc$, and $\perp$ is *falsum*.

5.1. Example. Transitivity may be given as the regular formula $(acb \wedge adc \wedge \neg adb) \rightarrow \perp$. Regular versions of totality, tower-symmetry, and n -height-restriction are $(\neg abc \wedge \neg acb) \rightarrow \perp$, $(acb \wedge \neg dba \wedge \neg dcb) \rightarrow \perp$, and $(\bigwedge_{i=1}^n \neg ab_{i+1}b_i) \rightarrow \perp$, respectively.

Clearly a regular formula $\varphi \rightarrow \perp$ translates directly into a pattern; just let the pattern comprise, in the obvious way, the set of conjuncts of φ . And if $\mathcal{C}_\Phi$ is the class of betweenness structures satisfying a given finite set Φ of regular formulas, then $\mathcal{C}_\Phi$ is also characterized via the corresponding set of patterns. Hence the following statement is a direct consequence of Theorem 4.1.

5.2. Corollary. *For $n = 1, 2, \dots$, the class of $\Lambda_n^\perp$ -induced betweenness structures is characterized by the following pattern set:*

$$\{\{acb, adc, \neg adb\}, \{\neg abc, \neg acb\}, \{acb, \neg dba, \neg dcb\}, \{\neg ab_{i+1}b_i : 1 \leq i \leq n\}\}.$$

Obtaining pattern sets corresponding to the characterization theorems of Section 3 is more involved because the axioms strong transitivity, extendability, and graphing do not readily translate into patterns.

By a **pseudo-regular** formula we mean the result of conjoining instances of ab and $a.b$ —but not their negations—to the premise of a regular formula. (For example, we express strong transitivity as the pseudo-regular formula $ab \wedge a.b \rightarrow \perp$.) We

⁹The image of f in $\langle X, I \rangle$ is a betweenness structure whose betweenness relation is sub-induced by I . As such it is not necessarily a substructure.

unpack a pseudo-regular formula $\varphi \rightarrow \perp$ into a set of patterns via the following steps.

- (1) Let V be a set of variables disjoint from A . We replace each occurrence of a formula of the form ab in φ with $(abv \vee vab)$, and each occurrence of $a.b$ with avb , where $v \in V$. We make sure that each replacement instance involves a new variable from V , and let the result of this replacement be φ' .
- (2) Rewrite φ' as a disjunction of conjunctions in the usual way, and call this formula φ'' .
- (3) Convert each disjunct of φ'' into its associated pattern over $A \cup V$; denote the set of these patterns by P_φ .

At this point, we have a set of patterns omitted by $\mathcal{C}_\varphi$. In order to obtain a characterization, we must expand P_φ by allowing variants of its constituent patterns. This we accomplish with the following step.

- (4) If $p \in P_\varphi$, then each $v \in V|p$ belongs to exactly one particle in p . For each $p \in P_\varphi$, an assignment $f : V|p \rightarrow (A \cup V)|p$ is **admissible** if p_f , the result of replacing each occurrence of variable $v \in V|p$ with the variable $f(v) \in (A \cup V)|p$, is a pattern (so no unintended duplications of variables from A). Note that if f is one-one onto $V|p$, then p_f is equivalent to p . Hence it makes sense at the outset to place an alphabetical (i.e., total) ordering $<$ on V such that we always have $f(v) \leq v$ whenever $f(v) \in V$. (This helps us eliminate redundancies. Note also that p_f may turn out to be self-contradictory, enabling us to pare down the resulting set of patterns.) Denote by $P_{\varphi,p}$ the set of patterns p_f for all such functions f . Since f can be the identity map, we have $p \in P_{\varphi,p}$ for each $p \in P_\varphi$. Let $P_\varphi^* := \bigcup \{P_{\varphi,p} : p \in P_\varphi\}$.

It is not difficult to show that the class $\mathcal{C}_\varphi$ of betweenness structures satisfying φ is characterized by the pattern set P_φ^* . However, the number of patterns in P_φ^* increases exponentially with the number of conjuncts of φ of the form ab . A mitigating factor is that the application of Step (4) leads to many patterns that are equivalent to one another, as well as patterns that are self-contradictory, allowing for the elimination of a fair amount of redundancy.

5.3. Examples. (i) Let φ be $ab \wedge a.b$, so that $\varphi \rightarrow \perp$ expresses the strong transitivity axiom. Then, after Step (1), we obtain $(abu \vee uab) \wedge avb$, and when we apply Step (2), we have $(abu \wedge avb) \vee (uab \wedge avb)$. Step (3) results in the pattern set $P_\varphi = \{\{abu, avb\}, \{uab, avb\}\}$, consisting of two patterns that are clearly equivalent: any realization of the first may be converted into a realization of the second by just transposing where the variables a and b are sent. Thus we may take P_φ to be $\{\{abu, avb\}\}$. When we apply Step (4)—noting that there are no admissible assignments that take values in $\{a, b\}$ —we have

$$P_\varphi^* = \{\{abu, avb\}, \{abu, aub\}\}, \text{there}$$

(We cannot shrink the set further: if we eliminate $\{abu, aub\}$, then a three-point betweenness structure will always omit $\{abu, avb\}$ by default, but three-point structures exist where strong transitivity fails; and if we eliminate $\{abu, avb\}$, then omitting $\{abu, aub\}$ alone amounts to satisfying antisymmetry.)

- (ii) Let Φ be $\{\varphi, \psi\}$, where φ is as in (i) and ψ is $ab \wedge ac \wedge b.c \wedge \neg bac$; so that $\psi \rightarrow \perp$ expresses the extendability axiom. Then, after Step (3) above, we have P_ψ comprising four patterns:

$$\begin{aligned} p_1 &= \{abu, acv, bwc, \neg bac\}; \\ p_2 &= \{abu, vac, bwc, \neg bac\}; \\ p_3 &= \{uab, acv, bwc, \neg bac\}; \text{ and} \\ p_4 &= \{uab, vac, bwc, \neg bac\}. \end{aligned}$$

Next we allow collapsing of the variables u, v, w within V , where $u < v < w$. This results in the collapses $\langle u, u, w \rangle$, $\langle u, v, u \rangle$, $\langle u, v, v \rangle$, and $\langle u, u, u \rangle$; hence each of these four patterns begets four more. For example, from p_1 we have $\{p_1, p_{11}, p_{12}, p_{13}, p_{14}\}$, where:

$$\begin{aligned} p_{11} &= \{abu, acu, bwc, \neg bac\}; \\ p_{12} &= \{abu, acv, buc, \neg bac\}; \\ p_{13} &= \{abu, acv, bvc, \neg bac\}; \text{ and} \\ p_{14} &= \{abu, acu, buc, \neg bac\}. \end{aligned}$$

So far we have twenty patterns (in addition to the two corresponding to φ), but we now need to allow for variables in $\{u, v, w\}$ to take on values from $\{a, b, c\}$. The admissible assignments for p_1 allow only $u \mapsto c$, $v \mapsto b$, or $w \mapsto a$. Also, if we allow $w \mapsto a$, we obtain both bac and $\neg bac$ as particles, leading to a self-contradictory pattern. If $u \mapsto c$ and $v \mapsto b$, then we obtain particles abc and acb in the same pattern, contradicting antisymmetry. (This pattern may be eliminated, as we are adding patterns to P_φ^* , which already deals with antisymmetry.) Hence we get just two more necessary patterns, $\{abc, acv, bwc, \neg bac\}$ and $\{abu, acb, bwc, \neg bac\}$, arising from p_1 alone.

For each of the patterns, p_{11}, p_{14} , there are no admissible assignments of variables from $\{u, v, w\}$ to variables from $\{a, b, c\}$. For p_{12} we have $\{abu, acb, buc, \neg bac\}$, and for p_{13} we have $\{abc, acv, bvc, \neg bac\}$. This gives a total of nine patterns ultimately arising from p_1 . A similar analysis gives seven patterns ultimately arising from each of p_2 and p_3 , and just five patterns ultimately arising from p_4 . In the end, P_Φ^* comprises: (1) the two patterns in P_φ^* ; and (2) the twenty-eight patterns ultimately arising from p_1, p_2, p_3, p_4 , for a total of thirty patterns.

- (iii) Let Ψ be $\Phi \cup \{\chi\}$, where Φ is as in (ii) and χ is

$$abc \wedge (ad \vee bd \vee cd) \wedge \neg adc \wedge \neg abd \wedge \neg bad \wedge \neg bcd \wedge \neg cbd;$$

so that $\chi \rightarrow \perp$ expresses the graphing axiom. This is equivalent to the conjunction $(\chi_a \rightarrow \perp) \wedge (\chi_b \rightarrow \perp) \wedge (\chi_c \rightarrow \perp)$, of three separate conditions, one for each of the premise disjuncts, ad , bd , and cd . Since each condition involves just one existential quantifier in its premise, we need only add one new symbol from V to our alphabet. Also, because of Lemma 3.17, we may delete the six-point patterns $\{p_1, p_2, p_3, p_4\}$ from P_Φ^* when we add in the patterns P_χ^* .

Re χ_a : After Step (3) we have P_{χ_a} comprising the two patterns

$$\begin{aligned} q_1 &= \{abc, adv, \neg adc, \neg abd, \neg bad, \neg bcd, \neg cbd\} \text{ and} \\ q_2 &= \{abc, vad, \neg adc, \neg abd, \neg bad, \neg bcd, \neg cbd\}. \end{aligned}$$

Since there is only one variable in V that we are using, the only new variants occur when we do the admissible substitutions $v \mapsto b, c$. The substitutions $v \mapsto c$ in q_1 and $v \mapsto b$ in q_2 result in self-contradictions; hence q_1 and q_2

beget the patterns $r_1 = \{abc, adb, \neg adc, \neg abd, \neg bad, \neg bcd, \neg cbd\}$ and $r_2 = \{abc, cad, \neg adc, \neg abd, \neg bad, \neg bcd, \neg cbd\}$, respectively. Thus we obtain four patterns from χ_a .

Re χ_b, χ_c : The analysis in each of these two cases is similar to—but not exactly the same as—that for χ_a , with the result that we obtain four patterns in each case. Hence P_χ^* comprises twelve patterns in all. Since there are thirty patterns in P_Φ^* and we may eliminate four of them in the presence of the graphing axiom, we conclude that P_Ψ^* comprises thirty-eight patterns.

With the discussion of Examples 5.3 (i,ii,iii), as well as the characterization theorems 3.5, 3.7, and 3.16, we have the following.

- 5.4. **Corollary.** (i) *Let $\mathcal{G}^-$ be the class of graph-subinduced (or M -subinduced) betweenness structures, with $\varphi \rightarrow \perp$ expressing the strong transitivity axiom. Then the pattern set P_φ^* (consisting of two patterns) characterizes $\mathcal{G}^-$.*
- (ii) *Let $\mathcal{M}$ be the class of M -induced betweenness structures, with $\psi \rightarrow \perp$ expressing the extendability axiom, and let $\Phi = \{\varphi, \psi\}$. Then the pattern set P_Φ^* (consisting of thirty patterns) characterizes $\mathcal{M}$.*
- (iii) *Let $\mathcal{G}$ be the class of graph-induced betweenness structures, with $\chi \rightarrow \perp$ expressing the graphing axiom, and let $\Psi = \{\varphi, \psi, \chi\}$. Then the pattern set P_Ψ^* (consisting of thirty-eight patterns), characterizes $\mathcal{G}$.*

6. TRIANGULAR OPERATIONS

Betweenness relations $+$ -induced or $\sqcup$ -induced in a metric space are specified via *triangle equalities* involving the binary operations $+$ and $\sqcup$, respectively, on $[0, \infty)$. We generalize this as follows. Define a binary operation $F : [0, \infty)^2 \rightarrow [0, \infty)$ to be **triangular** if for all $s, t \in [0, \infty)$ we have: (1) F is *commutative*, i.e., $F(s, t) = F(t, s)$; and (2) $s \sqcup t \leq F(s, t) \leq s + t$.

When F is a triangular operation, a metric ϱ on a set X is an **F -metric** if the *F -triangle inequality* holds; i.e., for all $x, y, z \in X$ we have

$$\varrho(x, y) \leq F(\varrho(x, z), \varrho(z, y)).$$

The pair $\langle X, \varrho \rangle$ is then called an **F -metric space**. (So $\sqcup$ -metric is synonymous with *ultrametric*.)

A triangular operation F is **associative** if it is associative in the usual sense of binary operations; i.e., if the equality

$$F(F(s, t), u) = F(s, F(t, u)) \text{ always holds.}$$

It is **monotone** if

$$F(s, t) \leq F(s', t') \text{ whenever } s \leq s' \text{ and } t \leq t',$$

and it is **sharply** (resp., **strongly**) **monotone** if, in addition,

$$F(s, t) < F(s', t') \text{ whenever } s < s' \text{ and } t < t' \text{ (resp., } s \leq s' \text{ and } t < t').$$

Sharp monotonicity and strong monotonicity are progressive strengthenings of monotonicity; we note that both $+$ and $\sqcup$ are triangular operations that are associative and sharply monotone, but only the former is strongly monotone.

Let $\mathcal{T}$ be the family of all triangular operations. For $F, G \in \mathcal{T}$, we write $F \leq G$ to mean $F(s, t) \leq G(s, t)$ for all $s, t \in [0, \infty)$. Thus F -metrics are automatically G -metrics when $F \leq G$, and $\leq$ makes $\mathcal{T}$ into a partially ordered set, with $\sqcup$ and $+$ as bottom and top elements, respectively.

If $F \in \mathcal{T}$ then $F(s, 0) = s$ for $s \in [0, \infty)$. We strictly order triangular operations by writing $F_0 < F_1$ if $F_0(s, t) < F_1(s, t)$ for all $s, t \in (0, \infty)$. In particular, $\sqcup(\cdot, \cdot) < +(\cdot, \cdot)$.

Given binary operations F_0 and F_1 on $[0, \infty)$, and $0 \leq \alpha \leq 1$, we define the *convex combination* F_α to be $(1 - \alpha)F_0 + \alpha F_1$, clearly another binary operation on $[0, \infty)$.

6.1. Lemma. *Suppose F_0 and F_1 are triangular operations and $0 \leq \alpha \leq 1$. Then the following are true.*

- (i) *Each convex combination F_α is also a triangular operation.*
- (ii) *If F_0 and F_1 are associative/monotone, then so is F_α .*
- (iii) *If F_0 is monotone, F_1 is sharply (resp., strongly) monotone, and $\alpha > 0$, then F_α is sharply (resp., strongly) monotone.*
- (iv) *If $F_0 \leq F_1$ and $\alpha < \beta$, then $F_\alpha \leq F_\beta$; if, in addition, $F_0 < F_1$, then $F_\alpha < F_\beta$.*

Proof. Ad (i). Clearly each F_α is commutative if both F_0 and F_1 are. Let $0 \leq \alpha \leq 1$ and $s, t \in [0, \infty)$. Then we have

$$(1 - \alpha)(s \sqcup t) \leq (1 - \alpha)F_0(s, t) \leq (1 - \alpha)(s + t) \text{ and} \\ \alpha(s \sqcup t) \leq \alpha F_1(s, t) \leq \alpha(s + t).$$

Adding the two inequalities gives

$$s \sqcup t \leq F_\alpha(s, t) \leq s + t.$$

Hence F_α is triangular.

Ad (ii). This is straightforward.

Ad (iii). Suppose F_0 is monotone, F_1 is sharply (resp., strongly) monotone, and $0 < \alpha \leq 1$. Let $0 \leq s < s'$ (resp., $0 \leq s \leq s'$) and $0 \leq t < t'$. Then we clearly have

$$F_\alpha(s, t) = (1 - \alpha)F_0(s, t) + \alpha F_1(s, t) < (1 - \alpha)F_0(s', t') + \alpha F_1(s', t') = F_\alpha(s', t').$$

Ad (iv). Suppose $F_0 \leq F_1$, $s, t \in [0, \infty)$, and $0 \leq \gamma < 1$. Then because $F_0(s, t) \leq F_1(s, t)$, we have

$$F_\gamma(s, t) = (1 - \gamma)F_0(s, t) + \gamma F_1(s, t) \leq (1 - \gamma)F_1(s, t) + \gamma F_1(s, t) = F_1(s, t);$$

hence $F_\gamma \leq F_1$.

Now assume $0 \leq \alpha < \beta \leq 1$, with $s, t \in [0, \infty)$. We let $G_0 = F_0$, $G_1 = F_\beta$, and for $0 \leq \gamma \leq 1$, we write $G_\gamma = (1 - \gamma)G_0 + \gamma G_1$. Then it is easy to check that $F_\alpha = G_{\alpha/\beta}$. Since $0 \leq \alpha/\beta < 1$, we know from the last paragraph that $G_{\alpha/\beta} \leq G_1$; i.e., that $F_\alpha \leq F_\beta$.

For the second assertion, we have a similar argument. Suppose $F_0 < F_1$, $s, t \in (0, \infty)$, and $0 \leq \gamma < 1$. Then because $F_0(s, t) < F_1(s, t)$ and $0 < 1 - \gamma$, we have

$$F_\gamma(s, t) = (1 - \gamma)F_0(s, t) + \gamma F_1(s, t) < (1 - \gamma)F_1(s, t) + \gamma F_1(s, t) = F_1(s, t);$$

hence $F_\gamma < F_1$.

So assume $0 \leq \alpha < \beta \leq 1$, with $s, t \in (0, \infty)$. We let $G_0 = F_0$, $G_1 = F_\beta$, and for $0 \leq \gamma \leq 1$, we write $G_\gamma = (1 - \gamma)G_0 + \gamma G_1$. Then, as before, we have $F_\alpha = G_{\alpha/\beta}$.

Since $0 \leq \alpha/\beta < 1$, we know from the last paragraph that $G_{\alpha/\beta} < G_1$; i.e., that $F_\alpha < F_\beta$. $\square$

6.2. Example. The partially ordered structure $\langle \mathcal{T}, \leq \rangle$ is not a total order. To see this, pick $\gamma > 0$ and define $H_\gamma(s, t) = \gamma$ if $s = t = \gamma$, and $H_\gamma(s, t) = s + t$ otherwise. Then clearly $H_\gamma \in \mathcal{T}$. However, if $\gamma \neq \delta$, then $H_\gamma(\gamma, \gamma) = \gamma < 2\gamma = H_\delta(\gamma, \gamma)$, while $H_\gamma(\delta, \delta) = 2\delta > \delta = H_\delta(\delta, \delta)$. Thus H_γ and H_δ are incomparable elements of $\mathcal{T}$. Recall that in the language of partial orderings, an *antichain* is a subset consisting of pairwise incomparable elements. Thus $\langle \mathcal{T}, \leq \rangle$ has antichains of cardinality $\mathfrak{c}$.

Note that no H_γ is continuous; its one point of discontinuity is $\langle \gamma, \gamma \rangle$. But with a small amount of extra work, one can produce two incomparable members of $\mathcal{T}$ that are continuous. Note also that any convex combination of two continuous triangular operations is continuous as well.

If F is a triangular operation and $\langle X, \varrho \rangle$ is any metric space, define the **F -induced ternary relation** I^F on X by stipulating that $I^F(x, z, y)$ is true precisely when the *F -triangle equality*

$$\varrho(x, y) = F(\varrho(x, z), \varrho(z, y)) \text{ holds.}$$

6.3. Proposition. *If F is a triangular operation, then every F -induced ternary relation is a betweenness relation.*

Proof. Symmetry holds because triangular operations are commutative. As for inclusivity: given $a, b \in X$, we wish to show $I^F(a, a, b)$ holds; i.e., that $\varrho(a, b) = F(\varrho(a, a), \varrho(a, b))$. But this is clearly true because the first argument in the right-hand side is zero. As for uniqueness: given $a, b \in X$, with $I^F(a, b, a)$ holding, we wish to show $a = b$. Indeed, we assume $0 = \varrho(a, a) = F(\varrho(a, b), \varrho(b, a))$. But the right-hand side dominates $\varrho(a, b) \sqcup \varrho(b, a) = \varrho(a, b)$; hence $\varrho(a, b) = 0$ and so $a = b$. $\square$

The following generalizes Proposition 2.1.

6.4. Proposition. *Let F and G be triangular operations, with $G < F$. If $\langle X, \varrho \rangle$ is a metric space and $a, b \in X$, we have $I^F(a, b) \cap I^G(a, b) = \{a, b\}$. Moreover, if the inducing metric is a G -metric then $I^F(a, b) = \{a, b\}$.*

Proof. If $c \notin \{a, b\}$, then $F(\varrho(a, c), \varrho(c, b)) > G(\varrho(a, c), \varrho(c, b))$. Hence no point other than a bracket point can lie in both intervals. If ϱ is a G -metric, then the F -triangle equality can never hold for points other than bracket points. $\square$

Let F and G be triangular operations. A betweenness structure $\langle X, I \rangle$ is **FG -induced** if I is F -induced by a G -metric. (We may also define the structure to be **FG -subinduced** the way we did in Section 3.)

From Section 2, $++$ -induced betweenness relations satisfy transitivity and antisymmetry, but not necessarily convexity; and $\sqcup\sqcup$ -induced relations satisfy convexity and totality, but not necessarily—indeed almost never—antisymmetry. Furthermore, $\sqcup+$ -induced betweenness need not satisfy transitivity or antisymmetry; while, by Proposition 6.4, FG -induced betweenness is trivial if $G < F$.

If $\mathcal{P}$ is a property of betweenness relations, let us call a triangular operation F **$\mathcal{P}$ -inducing** if every F -induced betweenness relation satisfies $\mathcal{P}$. We now consider the question of which triangular operations are $\mathcal{P}$ -inducing when $\mathcal{P}$ is one of the

first-order properties introduced above. We start with (weak) disjunctivity and totality.

6.5. Theorem. *The only triangular (resp., monotone triangular) operation that is totality-inducing (resp., weak disjunctivity-inducing) is $\sqcup$.*

Proof. The monotone triangular operation $\sqcup$ is totality-inducing and disjunctivity-inducing, by the results of Section 2.

Suppose $F \in \mathcal{T} \setminus \{\sqcup\}$. We may then fix $s, t \geq 0$ with $F(s, t) > s \sqcup t$. Let us assume $s \leq t$. Since $F(0, t) = t$, we have $0 < s \leq t < F(s, t)$.

To show F is not totality-inducing, let $X = \langle \{a, b, c\}, \varrho \rangle$, where $\varrho(a, b) = s$ and $\varrho(a, c) = \varrho(c, b) = t$. Then ϱ is a $\sqcup$ -metric; hence it is an F -metric. But $F(\varrho(a, b), \varrho(a, c)) = F(s, t) > t = \varrho(b, c)$, so it is not the case that $I^F(b, a, c)$ holds. By the same token, $I^F(a, b, c)$ fails to hold, showing that F is not totality-inducing.

Next we assume F is monotone and show it is not weak disjunctivity-inducing. By monotonicity, we have $0 < s \leq t < F(s, t) \leq F(t, t)$. Let $X = \langle \{a, b, c, d\}, \varrho \rangle$, where $\varrho(a, c) = \varrho(a, d) = \varrho(b, c) = \varrho(b, d) = t$ and $\varrho(a, b) = \varrho(c, d) = F(t, t)$. Then all four triangles of X are short isosceles, with the two short edges labeled t and the long edge labeled $F(t, t) > t$. The F -triangle inequality is clearly satisfied, we have both $I^F(a, c, b)$ and $I^F(a, d, b)$ holding but not $I^F(a, d, c)$ or $I^F(c, d, b)$ because $F(t, F(t, t)) \geq t \sqcup F(t, t) = F(t, t) > t$. Hence weak disjunctivity fails for $\langle X, I^F \rangle$. $\square$

We next consider transitivity and reversal.

6.6. Theorem. *Every associative monotone triangular operation is transitivity-inducing; if it is associative and strongly monotone, then it is reversal-inducing as well. In particular, $F_\alpha(s, t) := (1 - \alpha)(s \sqcup t) + \alpha(s + t)$ is transitivity-inducing for each $0 \leq \alpha \leq 1$, and reversal-inducing if and only if $0 < \alpha \leq 1$.*

Proof. Suppose F is an associative monotone triangular operation, with $\langle X, \varrho \rangle$ an F -metric space. Let $a, b, c, d \in X$ be such that $I^F(a, c, b)$ and $I^F(a, d, c)$ both hold. We first show that $I^F(a, d, b)$ also holds.

From the definition we have the equalities

$$(1) \varrho(a, b) = F(\varrho(a, c), \varrho(c, b))$$

$$(2) \varrho(a, c) = F(\varrho(a, d), \varrho(d, c))$$

both holding; hence, by substitution of (2) into (1) we have

$$(3) \varrho(a, b) = F(F(\varrho(a, d), \varrho(d, c)), \varrho(c, b)).$$

Applying associativity, we have

$$(4) \varrho(a, b) = F(\varrho(a, d), F(\varrho(d, c), \varrho(c, b))).$$

Applying monotonicity and the F -triangle inequality to the right-hand side of (4), we now have

$$(5) \varrho(a, b) \geq F(\varrho(a, d), \varrho(d, b)).$$

But, again by the F -triangle inequality, the right-hand side of (5) is $\geq \varrho(a, b)$; hence we have the equality

$$(6) \varrho(a, b) = F(\varrho(a, d), \varrho(d, b)),$$

and $I^F(a, d, b)$ holds as desired.

Now assume F is also strongly monotone. Then, from (4) and (6) we have

$$(7) F(\varrho(a, d), \varrho(d, b)) = F(\varrho(a, d), F(\varrho(d, c), \varrho(c, b))).$$

By strong monotonicity we now have

$$(8) \varrho(d, b) = F(\varrho(d, c), \varrho(c, b));$$

hence $I^F(d, c, b)$ holds.

The second assertion results from the first, in light of Lemma 6.1 and the remarks accompanying the definitions of *associative* and (*strongly*) *monotone* above. $\square$

6.7. Remark. By the time we get to equality (4) in the proof of Theorem 6.6, we have not yet used monotonicity; nor have we used the F -triangle inequality. Thus if F is an associative triangular operation, then any F +-induced betweenness structure satisfying reversal also satisfies transitivity.

6.8. Example. For $\gamma > 0$, let H_γ be the triangular operation from Example 6.2. Then:

(1) H_γ is not associative, as

$$H_\gamma(H_\gamma(\frac{\gamma}{2}, \frac{\gamma}{2}), \gamma) = H_\gamma(\gamma, \gamma) = \gamma, \text{ while}$$

$$H_\gamma(\frac{\gamma}{2}, H_\gamma(\frac{\gamma}{2}, \gamma)) = H_\gamma(\frac{\gamma}{2}, \frac{3\gamma}{2}) = 2\gamma.$$

(2) H_γ is not monotone, as

$$H_\gamma(\frac{\gamma}{2}, \gamma) = \frac{3\gamma}{2} > \gamma = H_\gamma(\gamma, \gamma). \text{ Finally,}$$

(3) H_γ is not transitivity-inducing; for let $X = \langle \{a, b, c, d\}, \varrho \rangle$, where

$$\varrho(a, b) = \varrho(b, c) = \varrho(a, c) = \varrho(d, b) = \gamma \text{ and}$$

$$\varrho(a, d) = \varrho(d, c) = \frac{\gamma}{2}.$$

All four triangles in X are isosceles; the only one that is not tall is the one with corners a, c, d , and one easily checks that the H_γ -triangle inequality holds here too. With $I = I^{H_\gamma}$, we see that both $I(a, c, b)$ and $I(a, d, c)$ hold, while $I(a, d, b)$ does not: $\varrho(a, b) = \gamma$, but

$$H_\gamma(\varrho(a, d), \varrho(d, b)) = H_\gamma(\frac{\gamma}{2}, \gamma) = \frac{3\gamma}{2}.$$

As to the question of which triangular operations are convexity-inducing, the answer is similar to the weak disjunctivity part of Theorem 6.5. Here, though, monotonicity gives way to sharp monotonicity.

6.9. Theorem. *The only sharply monotone triangular operation that is convexity-inducing is $\sqcup$.*

Proof. We already know $\sqcup$ is convexity-inducing (indeed every $\sqcup$ -induced betweenness relation is induced by a rank one road system, and we may cite Theorem 1.3.)

Suppose $F \in \mathcal{T} \setminus \{\sqcup\}$. Then, as in the weak disjunctivity part of the proof of Theorem 6.5, we have some $0 < t < F(t, t)$. Similar to that proof, let $X = \langle \{a, b, c, d, e\}, \varrho \rangle$, where $\varrho(a, b) = \varrho(c, d) = \varrho(a, e) = \varrho(b, e) = F(t, t)$ and $\varrho(a, c) = \varrho(a, d) = \varrho(b, c) = \varrho(b, d) = \varrho(c, e) = \varrho(d, e) = t$. Then the restriction of ϱ to $\{a, b, c, d\}$ is the four-point F -metric space from the proof of Theorem 6.5, and the edge labels of the six new triangles are easily checked to satisfy the F -triangle inequality. We observe that $I^F(a, c, b)$, $I^F(a, d, b)$, and $I^F(c, e, d)$ all hold, but that $I^F(a, e, b)$ does not: with the assumption that F is sharply monotone, and because

$t < F(t, t)$, we have $F(t, t) < F(F(t, t), F(t, t))$. Thus convexity does not hold for $\langle X, I^F \rangle$. $\square$

When we add Theorem 1.3 to the above, we obtain the following.

6.10. Corollary. *Let F be a sharply monotone triangular operation. Then $F = \sqcup$ if and only if every FF -induced betweenness structure is road-induced.*

We end this section with a consideration of when a triangular operation is antisymmetry-inducing. We already have sufficient conditions for the stronger property of reversal-inducing in Theorem 6.6, but antisymmetry—as a property expressed in first-order terms—is intrinsically simpler than reversal, involving three rather than four variables.

6.11. Theorem. *Let F be a triangular operation. Then F is antisymmetry-inducing if and only if $F > \sqcup$. In particular, $F_\alpha(s, t) := (1 - \alpha)(s \sqcup t) + \alpha(s + t)$ is antisymmetry-inducing if and only if $0 < \alpha \leq 1$.*

Proof. The condition that $F > \sqcup$ means that for all $s, t > 0$, we have $F(s, t) > s \sqcup t$. Suppose $F > \sqcup$ fails. Fix $0 < s \leq t$ such that $F(s, t) \leq t$. Then $F(s, t) = t$. Let $X = \langle \{a, b, c\}, \varrho \rangle$, where $\varrho(a, b) = s$ and $\varrho(a, c) = \varrho(b, c) = t$. Then X is an F -metric space where both $I^F(a, b, c)$ and $I^F(b, a, c)$ hold, but $a \neq b$. Hence X is an F -metric space that does not satisfy antisymmetry.

Suppose now that F is not antisymmetry-inducing. Then we have an F -metric space $\langle X, \varrho \rangle$ and a proper triple $\{a, b, c\}$ where $I^F(b, a, c)$ and $I^F(a, b, c)$ both hold. Suppose $s = \varrho(a, b)$, $t = \varrho(a, c)$, and $u = \varrho(b, c)$. Then we have: (1) $u = F(s, t)$; and (2) $t = F(s, u)$. From (1) we have $t \leq u$, and from (2) we have $u \leq t$, so $u = t$. Again from (1) we have $s \leq u$, so now $s \leq t$. Hence we have $F(s, t) = s \sqcup t$, showing that $F \not> \sqcup$.

The second statement follows from the first when we add in Lemma 6.1. $\square$

The following summarizes what we know about $\mathcal{P}$ -inducement for convex combinations of the extremal triangular operations.

6.12. Corollary. *For $0 \leq \alpha \leq 1$, let $F_\alpha(s, t) := (1 - \alpha)(s \sqcup t) + \alpha(s + t)$.*

- (i) F_α is transitivity-inducing.
- (ii) F_α is reversal-inducing if and only if it is antisymmetry-inducing, if and only if $\alpha > 0$.
- (iii) F_α is convexity-inducing if and only if it is weak disjunctivity-inducing, if and only if it is totality-inducing, if and only if $\alpha = 0$.

7. COMPATIBILITY OF BETWEENNESS RELATIONS AND METRICS

A triple $\langle X, I, \varrho \rangle$ is a **metric betweenness structure** if I is a betweenness relation on X and ϱ is a metric on X . The relation I is **compatible with ϱ** —or, equivalently, ϱ is compatible with I —if for each $a, b, c \in X$ such that $c \in I(a, b)$, we have $\varrho(a, c) \leq \varrho(a, b)$. If in addition we have $\varrho(a, c) < \varrho(a, b)$ for $c \in I(a, b)$ and $b \notin I(a, c)$, we say I is **strictly compatible with ϱ** .

7.1. Proposition. *Suppose $\langle X, I, \varrho \rangle$ is a metric betweenness structure where I is compatible with ϱ , $a \in X$, and $\langle a_n \rangle$ is a sequence of elements of X . If $\langle a_n \rangle$ ϱ -converges to a then $\bigcap_{n=0}^{\infty} I(a, a_n) = \{a\}$.*

Proof. Suppose $b \in \bigcap_{n=0}^{\infty} I(a, a_n)$ —which already contains a , by inclusivity. Then for each $n \in \omega$ we have, by compatibility, $\varrho(a, b) \leq \varrho(a, a_n)$. Since the right-hand quantity may be made arbitrarily small, owing to the convergence condition, we infer that $\varrho(a, b) = 0$ and hence $b = a$. $\square$

It is quite possible for $\bigcap_{n=0}^{\infty} I(a, a_n)$ to equal $\{a\}$ without $\langle a_n \rangle$ ϱ -converging to a ; just consider the $+$ -induced betweenness relation in the Euclidean plane. But what if we assume nestedness, i.e., where $I(a, a_0) \supseteq I(a, a_1) \supseteq \dots$? Assuming compatibility, we have $\varrho(a, a_0) \geq \varrho(a, a_1) \geq \dots \geq 0$, and can readily infer, by monotone convergence, that $\langle \varrho(a, a_n) \rangle$ converges to some $\alpha \geq 0$. This, however, is no assurance that $\langle a_n \rangle$ converges to a . With this in mind, define the betweenness relation I to be **convergence-compatible with** metric ϱ if it is compatible with ϱ and also satisfies the condition that whenever $I(a, a_0) \supseteq I(a, a_1) \supseteq I(a, a_2) \supseteq \dots$ is a nested sequence of intervals whose intersection is $\{a\}$, then the sequence $\langle a_n \rangle$ ϱ -converges to a .

7.2. Lemma. *Let $\langle X, I, \varrho \rangle$ be a metric betweenness structure where $I \subseteq I^F$ for some triangular operation F (so I is F +-subinduced by ϱ).*

- (i) *Then I is compatible with ϱ .*
- (ii) *If F is strongly monotone (e.g., $F = +$) then I is strictly compatible with ϱ .*
- (iii) *If ϱ is an F -metric (so I is FF -subinduced by ϱ) and $I = I^F$, then I is strictly compatible with ϱ .*
- (iv) *If F is continuous, $I = I^F$, and the ϱ -topology on X is compact, then I is convergence-compatible with ϱ .*

Proof. Ad (i). If $a, b, c \in X$ and $I(a, c, b)$ holds, then we have

$$\varrho(a, c) \leq \varrho(a, c) \sqcup \varrho(c, b) \leq F(\varrho(a, c), \varrho(c, b)) = \varrho(a, b).$$

Ad (ii). Assume $I(a, c, b) \wedge \neg I(a, b, c)$ holds. By (i), we know $\varrho(a, c) \leq \varrho(a, b)$. Since $b \notin I(a, c)$, we also know $b \neq c$; hence $\varrho(c, b) > 0$. By strong monotonicity, we have $F(s, \varrho(c, b)) > F(s, 0) = s$ for any $s \geq 0$. In particular, $F(\varrho(a, b), \varrho(c, b)) > \varrho(a, b)$. Hence $\varrho(a, c) \neq \varrho(a, b)$, and we have $\varrho(a, c) < \varrho(a, b)$.

Ad (iii). Assume ϱ is an F -metric, and that $I(a, c, b) \wedge \neg I(a, b, c)$ holds. Again we know $\varrho(a, c) \leq \varrho(a, b)$; suppose, for the sake of contradiction, we have equality. Since $I^F \subseteq I$, we have $\neg I^F(a, b, c)$; so by the F -triangle inequality, we know that $\varrho(a, c) < F(\varrho(a, b), \varrho(b, c))$. But then

$$\varrho(a, b) = \varrho(a, c) < F(\varrho(a, b), \varrho(b, c)) = F(\varrho(a, c), \varrho(c, b)) = \varrho(a, b),$$

a contradiction. Thus $\varrho(a, c) < \varrho(a, b)$ after all.

Ad (iv). Now suppose F is a continuous triangular operation, and assume $\langle X, \varrho \rangle$ is a compact metric space. Then because of the continuity of F , as well as of the metric, plus the assumption that $I = I^F$, we have that each

$$I(a, b) = \{x \in X : \varrho(a, b) - F(\varrho(a, x), \varrho(x, b)) = 0\}$$

is closed in X . Let $I(a, a_0) \supseteq I(a, a_1) \supseteq I(a, a_2) \supseteq \dots$ be a nested sequence of intervals, whose intersection is $\{a\}$. For the conclusion that $\langle a_n \rangle$ ϱ -converges to a , it suffices to show that each $B^\circ(a; r)$ contains some $I(a, a_m)$. Suppose otherwise, fix $r > 0$, and let $\mathcal{F}$ be the family $\{X \setminus B^\circ(a; r)\} \cup \{I(a, a_n) : n < \omega\}$. Then $\mathcal{F}$ is a family of closed subsets of $\langle X, \varrho \rangle$, a compact space, and $\mathcal{F}$ satisfies the

finite intersection property. Hence there is some $b \in \bigcap \mathcal{F}$. Since $b \neq a$, we have a contradiction. $\square$

Let $\langle X, I, \varrho \rangle$ be a metric betweenness structure. A point $a \in X$ is ***I*-isolated** if there exist $b \in X \setminus \{a\}$ and $r > 0$ such that $I(a, b) \cap B^\circ(a; r) = \{a\}$. (So a point being *I*-nonisolated means, roughly, that “in every direction, there are points arbitrarily close to the given point, but not equal to it.”) The metric betweenness structure is ***I*-discrete** if each of its points is *I*-isolated.

We first review some notions from model theory (see, e.g., [10]). Recall that a relational structure M (over a countable first-order language) is **ω_1 -saturated** if for every countable $A \subseteq M$ and set of formulas $\Phi(v)$ in a single free variable v in the language of M with additional constants from A , the following is true: $\Phi(v)$ is realized in M if (and only if) every finite subset of $\Phi(v)$ is realized in M . Every infinite ω_1 -saturated relational structure is uncountable, and every infinite relational structure has an ω_1 -saturated elementary extension. One way to obtain such an extension is to take an ultrapower of the original structure via a countably incomplete ultrafilter.

7.3. Lemma. *Let $\langle X, I, \varrho \rangle$ be a metric betweenness structure such that: (1) $\langle X, I \rangle$ is ω_1 -saturated; and (2) I satisfies transitivity and is compatible with ϱ . Then ϱ is *I*-discrete.*

Proof. We may assume X is infinite—at least nondegenerate—for otherwise ϱ is discrete and there is nothing to prove. Suppose ϱ is not *I*-discrete, and fix $a \in X$, a point that is *I*-nonisolated. Let $a_0 \in X \setminus \{a\}$ be arbitrary. Then there is some $a_1 \in (I(a, a_0) \cap B^\circ(a; 1)) \setminus \{a\}$. Now pick $a_2 \in (I(a, a_1) \cap B^\circ(a; 2^{-1})) \setminus \{a\}$. Continuing by induction, for each $n \geq 0$, we may find $a_{n+1} \in (I(a, a_n) \cap B^\circ(a; 2^{-n})) \setminus \{a\}$.

First we note that the sequence $\langle a_n \rangle$ ϱ -converges to a . Next, because transitivity holds for $\langle X, I \rangle$, we have $I(a, a_0) \supseteq I(a, a_1) \supseteq I(a, a_2) \dots$, a nested sequence of sets. Since I is ϱ -compatible, Proposition 7.1 tells us that $\bigcap_{n=0}^\infty I(a, a_n) = \{a\}$.

For $n \in \omega$, let $\varphi_n(v)$ be the first-order formula $I(a, v, a_n) \wedge v \neq a$. If $\Phi(v) = \{\varphi_n(v) : n \in \omega\}$, then, because of nestedness, a_{n+1} realizes the finite subset $\{\varphi_m(v) : 0 \leq m \leq n\}$ of $\Phi(v)$. Hence every finite subset of $\Phi(v)$ is realized in $\langle X, I \rangle$.

But $\langle X, I \rangle$ is ω_1 -saturated. Thus there is some $b \in X$ that realizes $\Phi(v)$; meaning that $b \neq a$ and $b \in \bigcap_{n=0}^\infty I(a, a_n)$. This contradiction completes the proof. $\square$

By combining Lemma 7.3 with some classical model theory, we have the following.

7.4. Theorem. *Let $\langle X, I \rangle$ be an infinite betweenness structure such that $\langle X, I \rangle$ satisfies transitivity. Then there is a betweenness structure $\langle X^*, I^* \rangle$ such that: (1) $\langle X^*, I^* \rangle$ is elementarily equivalent to $\langle X, I \rangle$; and (2) every metric on X^* compatible with I^* is I^* -discrete.*

Proof. By the Löwenheim-Skolem (upward) theorem, since X is infinite, $\langle X^*, I^* \rangle$ may be chosen to be an elementary extension of $\langle X, I \rangle$ of arbitrarily large infinite cardinality. Coupling this with the basic theory of ultrapowers, $\langle X^*, I^* \rangle$ may also be taken to be ω_1 -saturated (and therefore of cardinality at least $\aleph_1$). Since transitivity is a first-order property of betweenness structures, $\langle X^*, I^* \rangle$ satisfies it; hence we have our conclusion by Lemma 7.3. $\square$

As in Section 4, a set $C \subseteq X$ in a betweenness structure is an ***a*-chain** if for each two distinct $x, y \in C$ we have either $x <_a y$ or $y <_a x$. Then C is a **chain** if it is an *a*-chain for some $a \in X$.

The following extends Proposition 1.2.

7.5. Proposition. *Suppose $\langle X, I, \varrho \rangle$ is a metric betweenness structure, with value set $V \subseteq [0, \infty)$. If I satisfies transitivity and I is strictly compatible with ϱ —in particular if I is transitive and $+$ -subinduced by ϱ —then each chain in X has cardinality $\leq |V| \leq \mathfrak{c}$.*

Proof. Let $C \subseteq X$ be an a -chain for some $a \in X$. As in the proof of Proposition 1.2, define $f : C \rightarrow \mathbb{R}$ by $f(x) = \varrho(a, x)$. If $x, y \in C$ are distinct, say $x <_a y$, then, by the definition of *strictly compatible*, we know that $\varrho(a, x) < \varrho(a, y)$. This says that f is an injection. In the case where I is $+$ -subinduced by ϱ , we may invoke Lemma 7.2 (ii). $\square$

When we put Lemma 7.2 (iii) and (the first clause of) Theorem 6.6 together with Proposition 7.5, we immediately obtain the following.

7.6. Corollary. *Suppose $\langle X, I, \varrho \rangle$ is a metric betweenness structure, with value set $V \subseteq [0, \infty)$. If ϱ is an F -metric, where F is an associative monotone triangular operation, and $I = I^F$, then each chain in X has cardinality $\leq |V| \leq \mathfrak{c}$.*

7.7. Theorem. *Let $\langle X, I \rangle$ be a betweenness structure such that $\langle X, I \rangle$ satisfies transitivity and is compatible with a metric which is not I -discrete. Then there is a betweenness structure $\langle X^*, I^* \rangle$ such that: (1) $\langle X^*, I^* \rangle$ is elementarily equivalent to $\langle X, I \rangle$; and (2) I^* is not strictly compatible with any metric on X^* .*

Proof. Let I be compatible with a metric ϱ which is not I -discrete, and let $a \in X$ be a fixed I -nonisolated point. As in the proof of Lemma 7.3, we use an induction argument to create a sequence $\langle a_n \rangle$ which ϱ -converges to a , and where each a_{n+1} is contained in $I(a, a_n) \setminus \{a\}$. Furthermore, by stipulating that $\varrho(a_{n+1}, a) < \varrho(a_n, a)$, we can ensure that $a_n \notin I(a, a_{n+1})$ for any $n \in \omega$. Then, because of transitivity, we obtain an infinite chain $C = \{a_0, a_1, \dots\}$ in X .

Now create the relational structure $\langle X, I, C \rangle$, where C doubles as a unary predicate that is totally ordered by I in such a way that whenever $x, y \in C$ are distinct, then either $x <_a y$ or $y <_a x$ (as in the chain notation introduced above). Using the fact that C is an infinite chain, we now invoke the Löwenheim-Skolem theorem to obtain an elementary extension $\langle X^*, I^*, C^* \rangle$ of $\langle X, I, C \rangle$, such that the chain C^* contains more than $\mathfrak{c}$ elements of X^* . Since $\langle X^*, I^* \rangle$ also satisfies transitivity, Proposition 7.5 may be cited to obtain the desired conclusion. $\square$

Because elementary extensions can be simultaneously ω_1 -saturated and as large as we like, we can combine Theorems 7.4 and 7.7.

7.8. Corollary. *Let $\langle X, I \rangle$ be a betweenness structure such that $\langle X, I \rangle$ satisfies transitivity and is compatible with a metric which is not I -discrete. Then there is a betweenness structure $\langle X^*, I^* \rangle$ such that: (1) $\langle X^*, I^* \rangle$ is elementarily equivalent to $\langle X, I \rangle$; (2) every metric on X^* compatible with I^* is I^* -discrete; and (3) no metric on X^* is strictly compatible with I^* .*

7.9. Remarks. (i) Corollary 7.8 may be readily used to show that the class of FF -induced betweenness structures is not an elementary class when F is associative and monotone. This is also pointed out in [12] for the $++$ -case; their justification does not easily transfer to one for $\sqcup\sqcup$ -induced betweenness, however.

- (ii) The authors of [12] do get positive results if the notion of *metric* is broadened to allow distance values in an arbitrary ordered field. In this new sense the class of $++$ -induced betweenness structures is elementary; indeed it is axiomatizable via a recursive set of universal sentences. It is not finitely axiomatizable, however.

8. OTHER SOURCES OF RANK ONE FAMILIES

If T is a set and $\mathcal{N}$ is a nested family of subsets of T , then $\mathcal{R} = \mathcal{N} \cup \{\{t\} : t \in T\}$ is a rank one road system whose induced betweenness relation may or may not be $\sqcup\sqcup$ -induced.

8.1. Example. Let $T = \langle T, \preceq \rangle$ be a totally ordered set, with maximal element b , and with $\mathcal{R}$ the rank one road system consisting of the singletons of T , along with sets of the form $\{x \in T : t \preceq x\}$, where $t \in T$. Then T itself is a b -chain relative to the induced betweenness relation I ; hence, by Corollary 7.6, a necessary condition for I to be $\sqcup\sqcup$ -induced is for the cardinality of T to be $\leq \mathfrak{c}$.

In Example 2.7, the space $X = S^\omega$ is the set of *branches* (i.e., maximal chains) of the *tree* S^* of finite sequences from S , where $a \leq b$ in S^* if a is an initial segment of b . More generally, branch sets of trees (as discussed in [1]) give us another source of rank one road systems that do not depend on ultrametrics.

8.2. Example. If $\langle Y, \leq \rangle$ is a tree ordering—so a partial ordering with a unique minimal element, and such that the predecessors of each element form a chain—let X be the branch set of Y . For $\alpha, \beta, \gamma \in X$, define $I(\alpha, \gamma, \beta)$ to hold precisely when $\alpha \cap \gamma \supseteq \alpha \cap \beta$ (equivalently, $\gamma \cap \beta \supseteq \alpha \cap \beta$). In this way, the interval $I(\alpha, \beta)$ consists of all branches that “coalesce” with α and β above where α and β “coalesce” with each other. For each $a \in Y$, let $V(a) := \{\alpha \in X : a \in \alpha\}$. Then $\{V(a) : a \in Y\} \cup \{\{\alpha\} : \alpha \in X\}$ constitutes a rank one road system inducing I .

While the betweenness relation for the branch set in Example 2.7 is $\sqcup\sqcup$ -induced, we still need all the α -chains to have cardinality $\leq \mathfrak{c}$ for the same to be said of I in general. For a simple example, start with a bounded total ordering $\langle T, \preceq \rangle$, with a and b being the minimal and maximal elements, respectively. We then take $Y = T \times \{0, 1\}$, and define $\leq$ on Y as follows, where $s, t \in T$: (1) $\langle s, 0 \rangle \leq \langle t, 0 \rangle$ if and only if $s \preceq t$; (2) $\langle t, 0 \rangle \leq \langle t, 1 \rangle$; and (3) $\langle s, 1 \rangle \leq \langle t, 1 \rangle$ if and only if $s = t$. This makes X a tree with least element $\langle a, 0 \rangle$; and the proper $\leq$ -predecessors of each maximal element $\langle t, 1 \rangle$ are the $\leq$ -predecessors of $\langle t, 0 \rangle$. With α the branch in Y terminating at $\langle b, 1 \rangle$, we see that the branches terminating in the various points $\langle t, 1 \rangle$, $t \in T$, constitute an α -chain in X that has cardinality equal to that of T , which we may make as large as we like.

In [6, 7] we explore a topological form of betweenness, where if X is a continuum (i.e., a connected compact Hausdorff space) and $a, b \in X$, then the **subcontinuum interval** bracketed by $\{a, b\}$ is the intersection of all subcontinua of X that contain $\{a, b\}$. This makes the family $\mathcal{K}(X)$ of subcontinua of X a road system—an additive one at that—inducing subcontinuum betweenness.

Recall that a continuum X is **decomposable** if it is the union of two of its proper subcontinua; it is **hereditarily indecomposable** if no subcontinuum is decomposable.¹⁰ Hence no two of its subcontinua can overlap without being nested, which says directly that $\mathcal{K}(X)$ is a rank one family. In Proposition 2.14 we see that the totality axiom isolates the ultrametrics, from among all metrics, in terms of $\sqcup$ -induced betweenness. The same axiom isolates the hereditarily indecomposable continua from among all continua; the following is a slight rephrasing of [7, Proposition 5.6].

8.3. Proposition. *Let X be a continuum, with I its subcontinuum betweenness relation. Then $\langle X, I \rangle$ satisfies totality if and only if $\mathcal{K}(X)$ is a rank one family; i.e., X is a hereditarily indecomposable continuum.*

We now consider an important first-order axiom that is not universal.

- (*Gap-freeness*) $\forall u \forall v \exists w (u = v \vee (w \neq u \wedge w \neq v \wedge I(u, w, v)))$.

This axiom, akin to the density condition for total orderings, says that no nondegenerate betweenness intervals are gaps.

8.4. Proposition. *Let $\langle X, I \rangle$ be the subcontinuum betweenness structure arising from a hereditarily indecomposable continuum topology on X . Then $\langle X, I \rangle$ satisfies gap-freeness.*

Proof. If X is a hereditarily indecomposable continuum and $a, b \in X$, then the family of subcontinua of X containing $\{a, b\}$ is nested. Since (see [14]) intersections of nested families of continua are also continua, we see that $I(a, b)$ is the smallest subcontinuum containing $\{a, b\}$. Thus if a, b are distinct, then $I(a, b)$ is infinite. $\square$

If $\langle X, \varrho \rangle$ is a metric space, then I^+ can have gaps, even if the space has no isolated points. Any circle in the Euclidean plane, with the inherited Euclidean metric, has trivial $+$ -induced betweenness; also, By Proposition 2.1, I^+ is trivial if ϱ is an ultrametric.

8.5. Proposition. *Let $\langle X, I \rangle$ be the $\sqcup$ -induced betweenness structure arising from an ultrametric on X , where no point is isolated. Then $\langle X, I \rangle$ satisfies gap-freeness.*

Proof. Let $\langle X, \varrho \rangle$ be an ultrametric space, with $a, b \in X$ distinct. Since a is a nonisolated point, there is some $c \in X \setminus \{a\}$ such that $\varrho(a, c) < \varrho(a, b)$. Thus, since ϱ is an ultrametric, Lemma 2.4 tells us that $c \in I(a, b)$. $\square$

We next look at the existence of gaps in finite ultrametric spaces.

8.6. Theorem. *Every finite ultrametric space isometrically embeds in a finite ultrametric space that satisfies gap-freeness.*

Proof. Let $\langle X, \varrho \rangle$ be a finite ultrametric space, and assume $\{a, b\}$ is a gap. Then for each $c \in X \setminus \{a, b\}$, it is the case that $\varrho(a, b) < \varrho(a, c) = \varrho(b, c)$ (Lemma 2.4). We now let $X' = X \cup \{p\}$, where p is a new point not in X , and define the distance function ϱ' on X' by setting $\varrho'(x, y) := \varrho(x, y)$ if $x, y \in X$, by setting $\varrho'(p, a) = \varrho'(p, b) := \varrho(a, b)$, and by setting $\varrho'(p, c) := \varrho(a, c) (= \varrho(b, c))$ for $c \in X \setminus \{a, b\}$. We first show: (1) that ϱ' is an ultrametric on X' extending ϱ ; (2) that $\{a, b\}$ is not a gap in $\langle X', \varrho' \rangle$; and (3) that there are no gaps in $\langle X', \varrho' \rangle$ not already in $\langle X, \varrho \rangle$.

¹⁰Hereditarily indecomposable continua, though tricky to describe, are quite common (see [14]).

For notational convenience, let us abbreviate $\varrho'(x, y)$ by xy , for $x, y \in X'$. To prove ϱ' is an ultrametric, it suffices to look at the triangles in X' . All the ones in X are either tall isosceles or equilateral since ϱ is an ultrametric, so we examine triangles of the form $\{p, x, y\}$, where $x, y \in X$. We consider three cases.

Case 1. Suppose $\{x, y\} = \{a, b\}$. Then we have $px = py = xy$, and $\{p, x, y\}$ is equilateral.

Case 2. Suppose $x = a$ and $y \in X \setminus \{a, b\}$. Then $px = pa = ab < ay$ and thus $py = ay$. Thus $\{p, x, y\}$ is tall isosceles with short edge $\{p, x\}$.

Case 3. Suppose $\{x, y\} \cap \{a, b\} = \emptyset$. If $xy < ax$, then $by = bx = ax = px = py$, and $\{p, x, y\}$ is tall isosceles with short edge $\{x, y\}$. If $xy > ax = bx$, then $xy = by = py$ and $px = ax$. Thus $\{p, x, y\}$ is tall isosceles with short edge $\{p, x\}$. Finally consider the subcase where $xy = ax$. Then $ay = by = py \leq ax = px = xy$. Thus $\{p, x, y\}$ is either tall isosceles with short edge $\{p, y\}$, or it is equilateral. This shows that ϱ' is an ultrametric extending ϱ .

Clearly $\{a, b\}$ is not a gap in $\langle X', \varrho' \rangle$ since $p \in I^\perp(a, b) \setminus \{a, b\}$. There are no new gaps in $\langle X', \varrho' \rangle$: $\{p, a\}$ and $\{p, b\}$ are not gaps since their intervals contain b and a , respectively; if $x \in X \setminus \{a, b\}$, then $\{p, x\}$ is not a gap since it contains $\{a, b\}$.

The final step is to note that, because X has only finitely many gaps and the number of gaps in $\langle X', \varrho' \rangle$ is one less than the number of gaps in $\langle X, \varrho \rangle$, we may repeat the process above until there are no more gaps to consider. $\square$

We can continue the proof method of Theorem 8.6 in the following.

8.7. Theorem. *Every finite ultrametric space isometrically embeds in a countably infinite ultrametric space that satisfies gap-freeness.*

Proof. We first note that if $\langle X, \varrho \rangle$ is a finite ultrametric space with no gaps, then we can properly extend it to a finite ultrametric space $\langle X', \varrho' \rangle$ that has a gap as follows: Let p, q be two new points and define $\varrho'(p, a) := m$, where $m > 0$ is strictly smaller than $\varrho(x, y)$ for any distinct $x, y \in X$. Next define, for any $x \in X$, $\varrho'(p, x) = \varrho(q, x) := M$, where M is strictly larger than $\varrho(x, y)$ for any distinct $x, y \in X$. Then arguments similar to those in the proof of Theorem 8.6 show that ϱ' is an ultrametric on X' , strictly extending the ultrametric ϱ on X . Moreover, $\{p, q\}$ is a gap in $\langle X', \varrho' \rangle$.

So let $\langle X_0, \varrho_0 \rangle$ be our original finite ultrametric space. If there are no gaps, then introduce one using the last paragraph; if there are gaps, use Theorem 8.6. Using both these methods as necessary, we may obtain a sequence $\langle X_0, \varrho_0 \rangle, \langle X_1, \varrho_1 \rangle, \langle X_2, \varrho_2 \rangle, \dots$, where each $\langle X_{n+1}, \varrho_{n+1} \rangle$ is a finite ultrametric space that satisfies gap-freeness and properly extends $\langle X_n, \varrho_n \rangle$. Let $X_\omega := \bigcup \{X_n : n < \omega\}$, and define $\varrho(x, y) := \varrho_n(x, y)$ whenever $x, y \in X_\omega$ both lie in X_n . It is easy to see that $\langle X, \varrho \rangle$, being a nested union of ultrametric spaces, is an ultrametric space that extends $\langle X_0, \varrho_0 \rangle$. It is countably infinite because the cardinalities of its constituents increase without bound. Finally, any gap in the union is a gap in some $\langle X_n, \varrho_n \rangle$; so $\langle X, \varrho \rangle$ satisfies gap-freeness. $\square$

8.8. Remarks. (i) Using the approach of [8], we may view each countably infinite ultrametric space as of the form $\langle \mathbb{Q}, \varrho \rangle$, where $\mathbb{Q}$ is the set of rational numbers and ϱ is now seen as a binary operation on $\mathbb{Q}$. Then $\langle \mathbb{Q}, \varrho \rangle$ is a point in the Tychonoff power $\mathbb{Q}^{(\mathbb{Q}^2)}$, with $\mathbb{Q}$ given the $\{0, 1\}$ -valued metric. The power space is then a complete metric space and we may treat properties of ultrametrics as subsets of that space. When we do this, we may

use Theorem 8.7 to show that the set of all gap-free ultrametric spaces is *residual* (i.e., contains the intersection of countably many dense open sets) in the space of all ultrametric spaces. This gives meaning, in the Baire category sense, to the assertion, “Almost every countable ultrametric space satisfies gap-freeness.”

- (ii) There is also a game-theoretic way in which one may assert that gap-freeness is a “ubiquitous” property of ultrametrics. The game involves two players, Bob and Alice. Bob goes first and describes a finite ultrametric space—represented as an edge-labeled complete graph *à la* Example 2.3 (iii)—and Alice extends Bob’s offering to a new finite ultrametric space by adding at least one point. Bob properly extend’s Alice’s offering in a similar way; and the alternation of plays goes on forever, resulting in a countably infinite ultrametric space X . Alice is trying to ensure that X satisfies gap-freeness, and Bob is trying to force a gap. By Theorem 8.7, Alice has a winning strategy.

From Proposition 2.6 and Example 2.7, we know that a $\sqcup\sqcup$ -induced betweenness relation: (1) has no tall isosceles triangles if and only if the metric is two-valued; and (2) has no equilateral triangles if it is the ultrametric space of ω -sequences from a two-point set. The corresponding issue for hereditarily indecomposable continua is a bit less elementary: if $\langle X, I \rangle$ is the betweenness structure induced by the subcontinua of a hereditarily indecomposable continuum topology on X , then $\langle X, I \rangle$ has many triangles of both kinds.

Recall (see, e.g., [14]) that if X is a continuum (not necessarily metrizable) and $a \in X$, then the **composant** of X containing a is the set of points $b \in X$ such that $\{a, b\} \subseteq K$ for some proper subcontinuum K . It is well known [14] that if X is indecomposable, then the composants of X partition X into dense subsets. Moreover, if X is nondegenerate and metrizable, then the number of composants of X is $\mathfrak{c}$. If X is nondegenerate and nonmetrizable, however, the number of composants can be one, or any finite number. Even if the continuum is hereditarily indecomposable, the number of composants can be any $2 \leq n < \omega$. (See [15]: it is an open question whether a nondegenerate hereditarily indecomposable continuum can have just one composant.) Clearly if three points of an indecomposable continuum lie in three separate composants, then they form an equilateral triangle. Hence the number of composants has a bearing on the issue of triangles.

- 8.9. Lemma.** (i) ([14, Corollary 5.5] “boundary bumping”) *Let X be a continuum, with K a proper subcontinuum and U an open subset containing K . Then there is a subcontinuum $L \subseteq U$ that properly contains K .*
- (ii) ([9, Corollary 5]) *Every nondegenerate indecomposable continuum contains an indecomposable continuum with at least $\mathfrak{c}$ composants.*

Using these results, we have the following.

8.10. Proposition. *Let $\langle X, I \rangle$ be the subcontinuum betweenness structure arising from a hereditarily indecomposable continuum topology on X , with U a nonempty open subset of X .*

- (i) *X contains an equilateral triangle whose vertices lie in U .*
- (ii) *If $a \in U$ and $c \in X \setminus U$, then there is some $b \in U$ such that $\{a, b, c\}$ forms a tall isosceles triangle with short edge $\{a, b\}$.*

Proof. Ad (i): By Lemma 8.9 (i), we can find a nondegenerate subcontinuum $K \subseteq U$, and by Lemma 8.9 (ii), there is a subcontinuum $L \subseteq K$ such that L has uncountably many composants. All we need is three; so let $a, b, c \in L$ lie in separate composants of L . Then $I(a, b) = I(b, c) = I(a, c) = L$; hence $\{a, b, c\}$ is an equilateral triangle lying in U .

Ad (ii). Let $a \in U$ and $c \in X \setminus U$. By Lemma 8.9 (i), there is a nondegenerate subcontinuum $K \subseteq U$ containing a . For any $b \in K \setminus \{a\}$, the triangle $\{a, b, c\}$ is tall isosceles, with $\{a, b\}$ its short edge. $\square$

8.11. Remark. Boundary bumping (Lemma 8.9 (i)) allows us to take K to be a singleton, with U a cofinite set containing K . This special case may be expressed in a first-order manner—not as a single axiom but as an axiom schema, which we call *exclusivity*. For each $1 \leq n < \omega$, the n th axiom in the schema is the following.

$$\bullet \forall u_1 \dots u_n \exists v w (v \neq w \wedge \bigwedge_{i=1}^n \neg I(v, u_i, w)).$$

Not only does subcontinuum betweenness in a nondegenerate continuum satisfy exclusivity, but so too does any betweenness structure $\langle X, I \rangle$, where I is compatible (in the sense of Section 7) with a metric ϱ that affords infinitely many nonisolated points of X . Indeed, let $F \subseteq X$ be finite. Then there is some $a \in X$ and $r > 0$ such that a is nonisolated and $B_\varrho^\circ(a; r) \cap F = \emptyset$. Pick $b \in B_\varrho^\circ(a; r) \setminus \{a\}$. Then $I(a, b) \subseteq B_\varrho^\circ(a; r)$, by compatibility, so $I(a, b)$ misses F .

We finish with a brief discussion of when the subcontinuum betweenness structure of a nondegenerate hereditarily indecomposable continuum is *compatible*—in various interpretations of the word—with a metric. If the metric generates the continuum topology, betweenness intervals can be unboundedly large compared to how close their bracket points are to each other; in other words, intervals can look quite “bendy” relative to the metric. We now introduce a natural method of *stretching* a given metric in such a way that betweenness intervals look *straighter* relative to this new metric. We say that a metric σ **topologically refines** metric ϱ on a set if every ϱ -open set is a σ -open set.

Given a metric continuum $\langle X, \varrho \rangle$ and $A \subseteq X$, we denote by $\delta(A) = \delta_\varrho(A)$ the ϱ -diameter of A , namely

$$\delta(A) := \sup\{\varrho(x, y) : x, y \in A\}.$$

Then $\delta(A)$ is always finite (because metrics generating compact topologies are bounded) and we define, for each $a, b \in X$ the **stretch distance** $\varrho^I(a, b)$ to equal $\delta(I(a, b))$, where I is the subcontinuum betweenness structure of X . With this stretching operation, the new distance between two points is at least as great as the old.

8.12. Theorem. *Let $\langle X, \varrho \rangle$ be a nondegenerate hereditarily indecomposable metric continuum, with I the subcontinuum betweenness structure of X . Then:*

- (i) I and ϱ are incompatible; and
- (ii) ϱ^I is an ultrametric that topologically refines ϱ and $\sqsubset$ -subinduces I .

Proof. It makes the most sense to prove the two statements of the theorem in reverse order.

Ad (ii). The stretch distance function clearly satisfies metric symmetry, and the obvious fact that $\varrho^I(a, b) \geq \varrho(a, b)$ always holds implies positivity, so we now show the $\sqcup$ -triangle inequality holds for $\varrho^{I'}$.

Since X is a hereditarily indecomposable continuum, the family of subcontinuum betweenness intervals—each itself a subcontinuum—is of rank one. Hence if $a, b, c \in X$ are arbitrary, we know that $\{I(a, c), I(c, b)\}$ is nested since the two sets overlap. Because of disjunctivity always holding for subcontinuum betweenness, we have $I(a, b) \subseteq I(a, c) \cup I(c, b)$; hence

$$\varrho^I(a, b) = \delta(I(a, b)) \leq \delta(I(a, c) \cup I(c, b)) = \varrho^I(a, c) \sqcup \varrho^I(c, b),$$

showing that ϱ^I is an ultrametric. Since ϱ^I -distances always dominate corresponding ϱ -distances, ϱ^I clearly refines ϱ topologically.

It remains to show $I \subseteq I_{\varrho^I}^{\sqcup}$. Indeed, fix $a, b, c \in X$, with $I(a, c, b)$ holding, so that $I(a, c) \cup I(c, b) \subseteq I(a, b)$. Again because $\{I(a, c), I(c, b)\}$ is nested, we have $\varrho^I(a, c) \sqcup \varrho^I(c, b) \leq \varrho^I(a, b)$; so equality holds since ϱ^I has already been shown to be an ultrametric. Thus $I_{\varrho^I}^{\sqcup}(a, c, b)$ holds.

Ad (i). If $\langle X, \varrho \rangle$ is any metric continuum, and if I is compatible with ϱ , fix $a, b \in X$. If $x, y \in I(a, b)$, then $\varrho(x, y) \leq \varrho(a, x) + \varrho(a, y) \leq 2\varrho(a, b)$. Hence we have $\varrho(a, b) \leq \varrho^I(a, b) \leq 2\varrho(a, b)$. But in the case where X is hereditarily indecomposable, we know from (ii) that ϱ^I is a *metric*, and so this inequality implies that ϱ and ϱ^I are topologically equivalent. But we also know from Remark 2.9, plus the fact that ϱ^I is an *ultrametric*, that ϱ^I generates a zero-dimensional topology and is manifestly not equivalent to ϱ . Hence I and ϱ are incompatible. $\square$

In the proof of Theorem 8.12 (ii) we were unable to show $I_{\varrho^I}^{\sqcup} \subseteq I$ (and therefore that $I_{\varrho^I}^{\sqcup} \sqcup$ -induces I). To do this we would need to prove that the diameter of $I(a, b)$ is strictly less than that of $I(a, c)$ just because the first set is properly contained in the second. Diameters almost never behave this way, however, and we currently do not know of a hereditarily indecomposable metric continuum whose stretch metric $\sqcup$ -induces the subcontinuum betweenness relation. We give an affirmative answer to a slightly weaker version of this question, as well as examine stretch metrics from a more general perspective, in a future article [2].

9. LOOSE ENDS

The following is a short list of questions that came up during the course of this investigation.

- 9.1. Questions.**
- (1) (From Section 3) In Theorem 3.1, is $\mathcal{B}(\Lambda^+)$ axiomatizable by a *finite* set of universal axioms?
 - (2) (From Section 3) What are some interesting betweenness properties that are $+$ -induced by the Rado graph?
 - (3) (From Section 6) Is there a first-order betweenness property $\mathcal{P}$ such that the only $\mathcal{P}$ -inducing triangular operation (possibly with other conditions added; e.g., monotonicity) is $+$?
 - (4) (From Section 8) Find betweenness properties that hold for all betweenness structures compatible with certain kinds of metrics. (For example, a betweenness structure satisfies exclusivity if it is compatible with a metric that admits infinitely many nonisolated points.)

- (5) (From Section 8) Under what circumstances is the subcontinuum betweenness relation of a hereditarily indecomposable metric continuum $\sqcup$ -induced by the stretch of its metric?

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DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, MARQUETTE UNIVERSITY, MILWAUKEE, USA

E-mail address: paul.bankston@marquette.edu

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF GALWAY, GALWAY, IRELAND

E-mail address: aisling.mccluskey@universityofgalway.ie

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, TORONTO, CANADA

E-mail address: swatson@yorku.ca