Betweenness and Equidistance in Metric Spaces

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Betweenness and equidistance are ternary relations on a set X, and—like ordering relations— arise throughout mathematics.

$I(a, c, b)$ is read, "c lies between a and b;"
$E(a, c, b)$ is read, "c lies equidistant from a and b."

We then write

$$I(a, b) := \{x \in X : I(a, x, b) \text{ holds}\},$$

the *interval bracketed by* a, b;

$$E(a, b) := \{x \in X : E(a, x, b) \text{ holds}\},$$

the *equiset with cocenters* a, b.

As for axioms, there does not seem to be as much consensus as there is in the case of–say–partial orderings. However the following seem to be the least controversial betweenness axioms, and hold in the major examples.

- (Inclusivity) $\{a, b\} \subseteq I(a, b)$.

- (Symmetry) $I(a, b) = I(b, a)$.

- (Uniqueness) $I(a, a) = \{a\}$.

- (Transitivity) If $c \in I(a, b)$, then $I(a, c) \subseteq I(a, b)$.
  ("$c \leq_a b$ & $d \leq_a c \implies d \leq_a b$")

In addition we have axioms that hold in some interesting cases, but not in others.

- (Antisymmetry) If $c \in I(a,b)$ and $b \in I(a,c)$, then $b = c$. ("$c \leq_a b$ & $b \leq_a c \implies b = c$")

- (Convexity) If $c, d \in I(a,b)$, then $I(c,d) \subseteq I(a,b)$.

Define a set $K \subseteq X$ to be I-*convex* if $I(a,b) \subseteq K$ for all $a, b \in K$. Then the convexity axiom above says that intervals themselves are I-convex. (SHOULDN'T THEY BE?)

Well, not always. Consider first betweenness in a metric space $\langle X, \varrho \rangle$, as introduced in the 1920s by Menger. The *metric interval* bracketed by a, b is defined as

$$I_\varrho(a,b) := \{x \in X : \varrho(a,b) = \varrho(a,x) + \varrho(x,b)\}.$$

(And the equiset is defined in the obvious way as

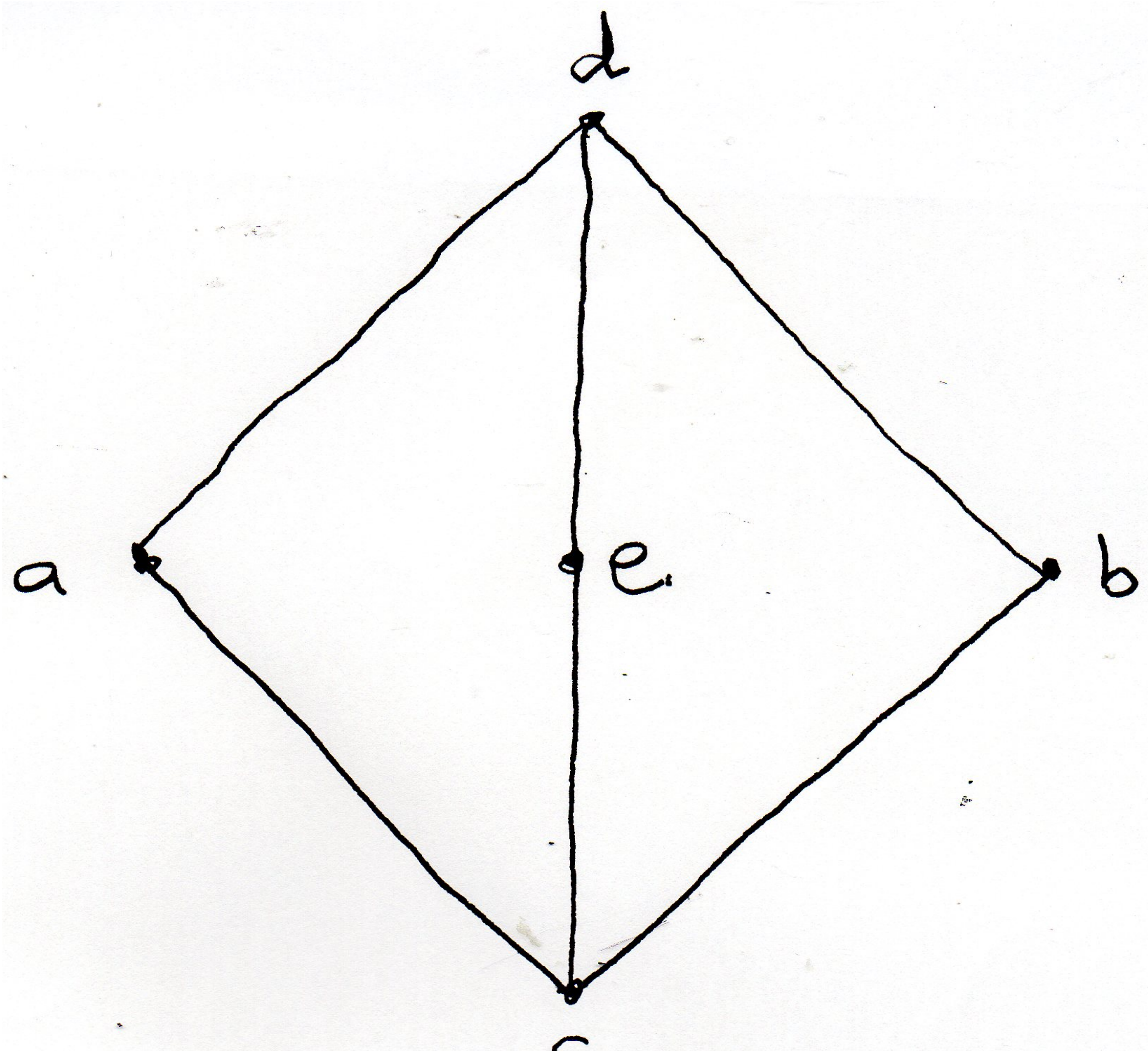
$$E_\varrho(a,b) := \{x \in X : \varrho(a,x) = \varrho(x,b)\}.)$$

It's easy to show this interpretation of betweenness satisfies all the axioms above, with the possible exception of convexity.

It is indeed possible for $c, d \in I_\varrho(a, b)$, with $I_\varrho(c, d) \not\subseteq I_\varrho(a, b)$. (Indeed, for metric betweenness, convexity is more common "in the breach than the observance.")

Here's an easy example. In $\mathbb{R}^2$ let X be the theta-curve as shown; the metric is given by minimal path length according to the Euclidean metric. Then $c, d \in I_\varrho(a, b)$ (= the outer square), but

$$e \in I_\varrho(c, d) \setminus I_\varrho(a, b).$$


d
a
e
b
c

If X is a vector space over the real scalar field, the *linear interval* bracketed by a, b is defined parametrically as

$$[\![a, b]\!] := \{(1 - t)a + tb : 0 \leq t \leq 1\},$$

with *linear betweenness* defined accordingly.

(This is otherwise known as the *line segment with end points* a *and* b.)

Linear betweenness in vector spaces satisfies all of the axioms above. And if $\langle X, \| \cdot \| \rangle$ is a normed vector space with the norm-induced metric, we have two independently-defined notions of betweenness, clearly related by the fact that

$$[\![a, b]\!] \subseteq I_{\varrho}(a, b)$$

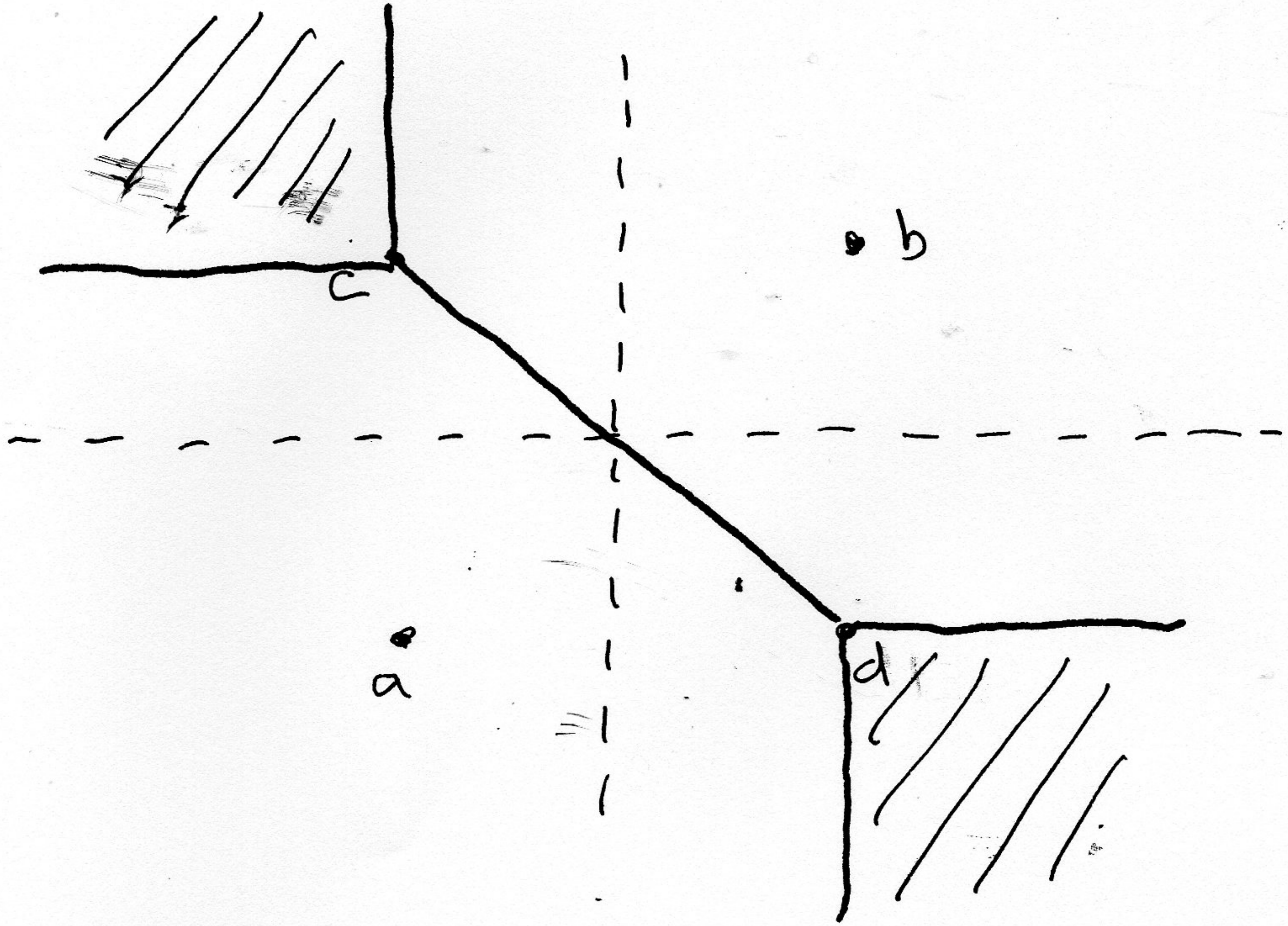
always holds. Generally the inclusion is proper; a metric interval is said to be *linear* if it equals its corresponding linear interval.

In Euclidean space, metric intervals are all linear, and the equiset cocentered at two distinct points is the hyperplane that is the perpendicular bisector of the line segment with those points as end points.

In the case of the Cartesian plane $\mathbb{R}^2$ equipped with the *taxicab* norm, the metric interval $I_\varrho(a, b)$ is the rectangle with $[\![a, b]\!]$ as a diagonal, and with sides parallel to the coordinate axes. In the case $a = \langle -1, -1\rangle$, $b = \langle 1, 1\rangle$, $c = \langle -1, 1\rangle$ and $d = \langle 1, -1\rangle$, $E(a, b)$ is the set

$$[\![c, d]\!] \cup ((-\infty, -1] \times [1, \infty)) \cup ([1, \infty) \times (-\infty, -1]),$$

a line segment joining two disjoint quarter-planes.


b
c
a
d

In any metric space X, each metric interval $I_\varrho(a,b)$ is closed because it is the zero set of the continuous mapping

$$x \mapsto \varrho(a,x) + \varrho(x,b) - \varrho(a,b);$$

it is bounded, as its diameter is $\varrho(a,b)$.

When the metric is norm-induced, metric intervals are compact in the finite-dimensional case, but not in general. The Banach space c_0, defined as the set of null sequences with the supremum norm, has the property that no nondegenerate metric interval is compact.

Each equiset $E_\varrho(a, b)$ is closed because it is the zero set of the map

$$x \mapsto \varrho(a, x) - \varrho(x, b).$$

In the norm-induced case, however, it is never bounded.

In a vector space X, a set is *linearly convex* if it is I-convex under the linear interpretation of betweenness; similarly we define a subset of a metric space to be *metrically convex*.

Linear convexity is just convexity in the classical sense; it is generally much weaker than metric convexity in the context of normed vector spaces.

1 Proposition. In a normed vector space, all metric intervals are linearly convex; however [Panda-Kapoor, 1974] equisets are linearly convex iff the norm is induced by an inner product.

In any metric space X, define the *closed ball* and *sphere* of radius $r \geq 0$ and center $a \in X$ by

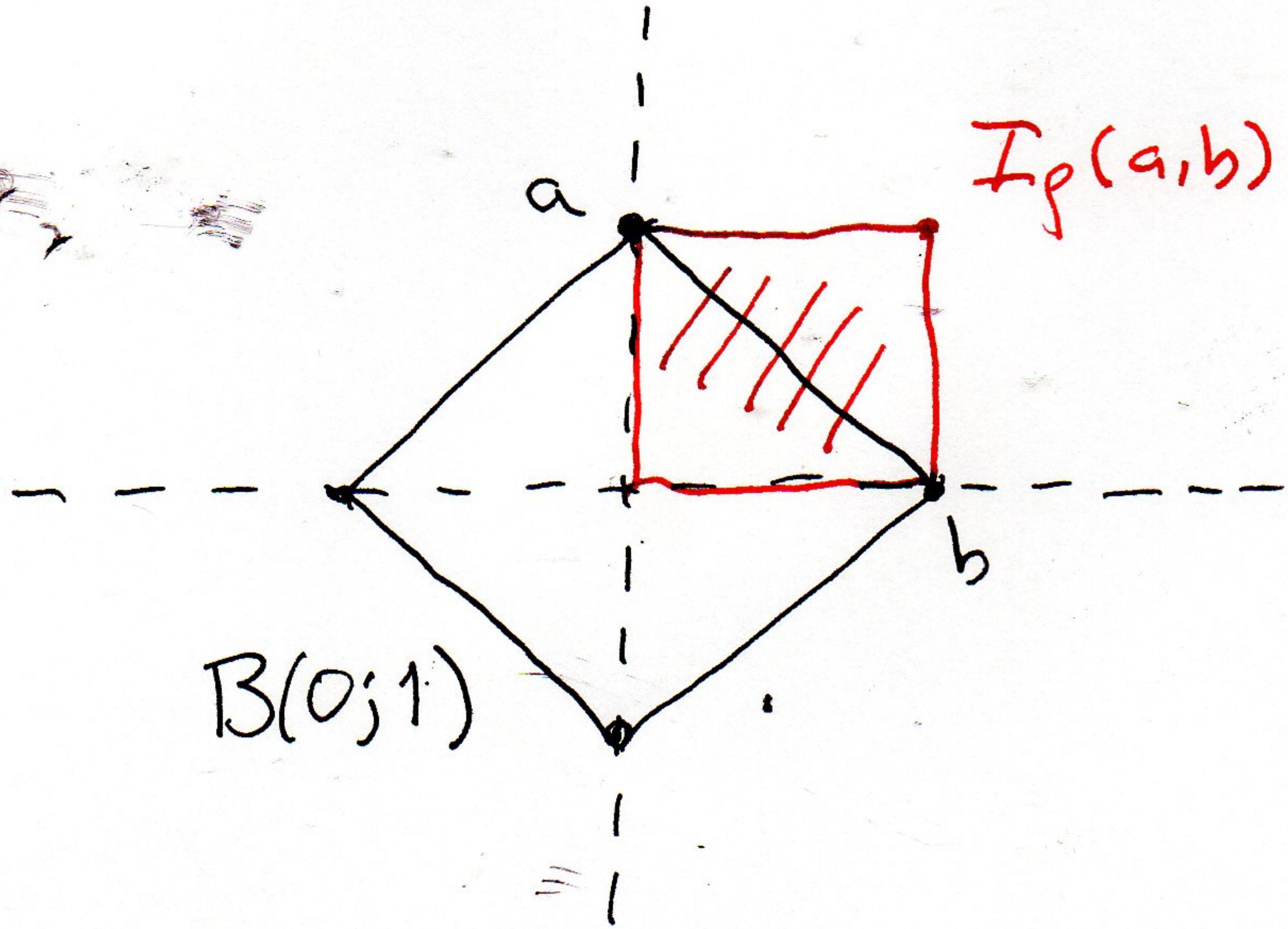
$$B(a; r) := \{x \in X : \varrho(x, a) \leq r\}$$

and

$$S(a; r) := \{x \in X : \varrho(x, a) = r\},$$

respectively.

Then closed balls in normed vector spaces are well known to be linearly convex; they can easily fail to be metrically convex. (Consider $\mathbb{R}^2$ under the taxicab norm.)



$I_\rho(a,b)$
a
b
B(0;1)

A point e in a linearly convex subset K of a vector space is an *extreme point* if e cannot be a relative interior point of a line segment that is contained in K. Let $\mathsf{E}(K)$ be the set of extreme points of K.

The Krein-Milman theorem states that *if K is a compact linearly convex subset of a normed vector space, then K is the closed linearly convex hull of* $\mathsf{E}(K)$.

Closed balls of positive radius in a normed vector space are compact iff the dimension of the space is finite. It is possible for closed balls to have no extreme points (viz. c_0 and the next proposition); at any rate it is the case that

$$\mathsf{E}(B(a;r)) \subseteq S(a;r).$$

2 Proposition Let X be a normed vector space, $a, b \in X$ distinct. Then $I_\varrho(a,b)$ is linear iff $\frac{a-b}{\|a-b\|}$ is an extreme point of $B(0;1)$.

A normed vector space is called *strictly convex* if each point of the unit sphere is an extreme point of the unit ball. (This is equivalent to the *rotundity* condition that if a, b are nonzero vectors such that $\|a+b\| = \|a\| + \|b\|$, then each of the vectors is a positive multiple of the other.

So, as a corollary to Proposition 2: *A normed space is strictly convex iff each of its metric intervals is linear.*

Let's now examine what metric intervals look like in the case of a normed plane.

It suffices to look at pairs $\langle a, b\rangle$ of points where $a = 0$ and $b \in S(0;1)$.

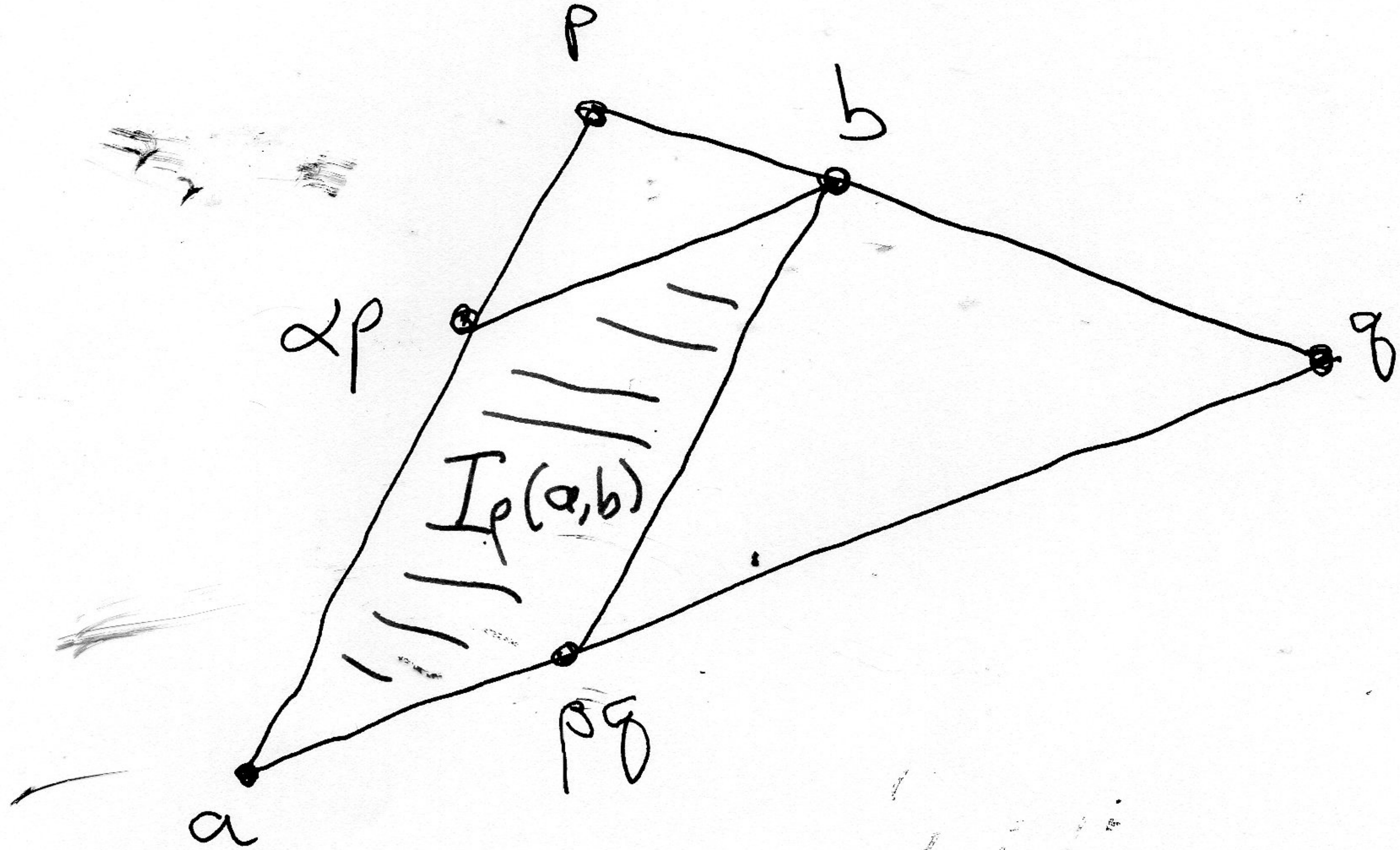
Step 1. If $b \in \mathsf{E}(B(0;1))$, then $I_\varrho(a,b) = [\![a,b]\!]$ (from Proposition 2).

Step 2. Otherwise, there exist distinct $p, q \in \mathsf{E}(B(0;1))$, and $\alpha, \beta > 0$ such that:

(2.1) $\alpha + \beta = 1$ and $b = \alpha p + \beta q$;

(2.2) $[\![p,q]\!] \cap \mathsf{E}(B(0;1)) = \{p,q\}$.

Step 3. Then $I_\varrho(a,b)$ is the parallelogram with vertices $0, \alpha p, \beta q, b$.


p
b
αp
q
I_p(a,b)
pq
a

3 Proposition. *All metric intervals in a normed plane are metrically convex. Somewhat more generally: if* X *is a normed vector space such that all the non-extreme points of the unit ball are coplanar, then all metric intervals of* X *are metrically convex.*

Let's now consider metric hyperspaces. For a metric space $\langle X, \varrho \rangle$, $K(X)$ denotes its hyperspace of nonempty compact subsets, and $F_1(X)$ its hyperspace of singletons.

When X is a normed vector space, $KL(X)$ is its hyperspace of linearly convex compact subsets. We assume all hyperspaces of X to lie between $F_1(X)$ and $K(X)$.

The *Hausdorff metric* $\varrho_H(\cdot,\cdot)$ is defined on $K(X)$ in three steps:

Step 1. For $a \in X$ and $B \in K(X)$,

$$\varrho(a, B) := \min\{\varrho(a, b) : b \in B\}$$

Step 2. For $A, B \in K(X)$,

$$\varrho(A, B) := \max\{\varrho(a, B) : a \in A\}$$

Step 3. For $A, B \in K(X)$,

$$\varrho_H(A, B) := \max\{\varrho(A, B), \varrho(B, A)\}$$

When we restrict our attention to $KL(X)$ in the normed situation, a consequence of the Bauer maximality principle is that we may replace Step 2 with

Step 2'. For $A, B \in KL(X)$,

$$\varrho(A,B) := \max\{\varrho(a,B) : a \in \mathsf{E}(A)\}$$

In particular,

$$\varrho_H(\{a\},B) = \max\{\varrho(a,b) : b \in \mathsf{E}(B)\}.$$

If $\mathcal{H}$ is a hyperspace of X and $Y \subseteq X$, we denote by $\mathcal{H}[Y]$ the restriction

$$\{A \in \mathcal{H} : A \subseteq Y\}.$$

We consider singleton brackets and cocenters.

For $A, B \in K(X)$ and $\mathcal{H}$ a hyperspace of X, $I_{\mathcal{H}}(A, B)$ is the hyperspace interval

$$\{C \in \mathcal{H} : \varrho_H(A, B) = \varrho_H(A, C) + \varrho_H(C, B)\}.$$

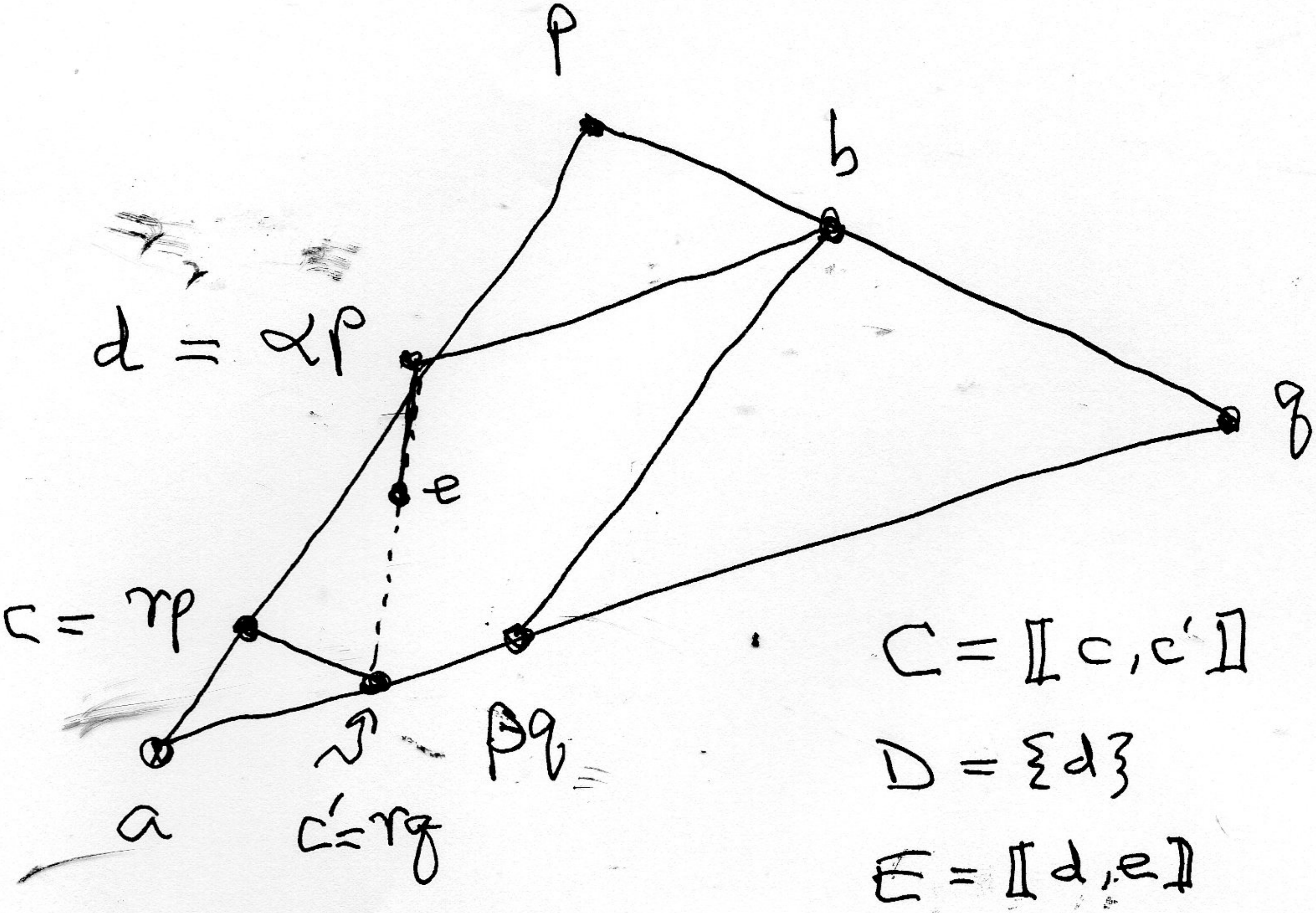
(Similarly for $E_{\mathcal{H}}(A, B)$.)

4 Proposition. *Let X be a metric space, with $a, b \in X$ and $\mathcal{H}$ a hyperspace of X. Then* $I_{\mathcal{H}}(\{a\},\{b\}) =$

$$\{C \in \mathcal{H} : C \subseteq I_{\varrho}(a,b) \ \&\ \forall x,y \in C, \varrho(a,x) = \varrho(a,y)\}$$

$$=$$

$$\bigcup\{\mathcal{H}[S(a;r) \cap I_{\varrho}(a,b)] : 0 \leq r \leq \varrho(a,b)\}.$$

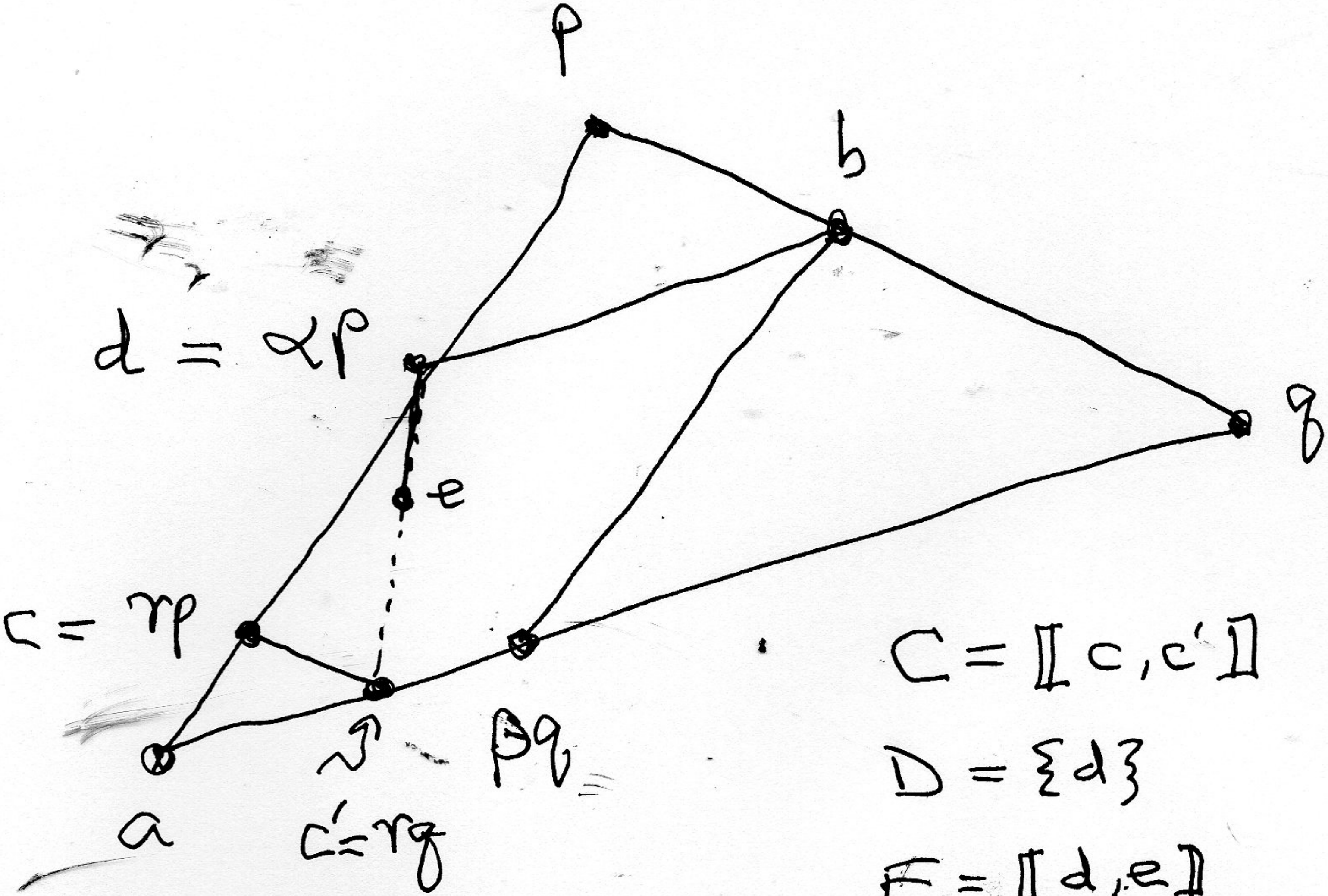

p
b
q
d = αp
e
c = γp
a
c′ = γq
βq
C = ⟦c, c′⟧
D = {d}
E = ⟦d, e⟧

For hyperspace equisets, we have a messy equality, which has the following succinct inequality as a corollary.

5 Proposition. *Let X be a metric space, with $a, b \in X$ and $\mathcal{H}$ a hyperspace of X. Then* $E_{\mathcal{H}}(\{a\}, \{b\}) \supseteq$

$$\mathcal{H}[E_{\varrho}(a,b)] \cup \{C \in \mathcal{H} : \{a,b\} \subseteq C \subseteq I_{\varrho}(a,b)\}.$$

6 Proposition. *Let X be a normed vector space, with $a, b \in X$. Then $I_{KL(X)}(\{a\}, \{b\})$ is metrically convex iff $I_{\varrho}(a, b)$ is linear.*


p
b
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d = αp
e
c = γp
a
c′ = γq
βq
C = ⟦c, c′⟧
D = {d}
E = ⟦d, e⟧

7 Corollary. *Let X be a normed vector space. Then X is strictly convex iff all metric intervals in $KL(X)$ with singleton brackets are metrically convex.*

Note that, because all metric intervals in a normed plane are metrically convex, strict convexity is a much stronger condition than having all metric intervals in X be metrically convex.

If X is a normed vector space and $A, B \in K(X)$, then define the *linear hyperinterval* bracketed by A, B to be

$$[\![A, B]\!] := \{(1-t)A + tB : 0 \leq t \leq 1\}.$$

This is a setwise convex combination of the two brackets, and is guaranteed to comprise a family of compact subsets of X.

However, it does not yield an acceptable notion of betweenness for $K(X)$, as it doesn't necessarily satisfy the uniqueness axiom: if A is the doubleton set $\{a, b\}$, then $[\![A, A]\!]$ contains the three-element set $\{a, b, \frac{1}{2}a + \frac{1}{2}b\}$, and is therefore not equal to $\{A\}$.

In particular, we cannot always conclude that

$$[\![A, B]\!] \subseteq I_{K(X)}(A, B).$$

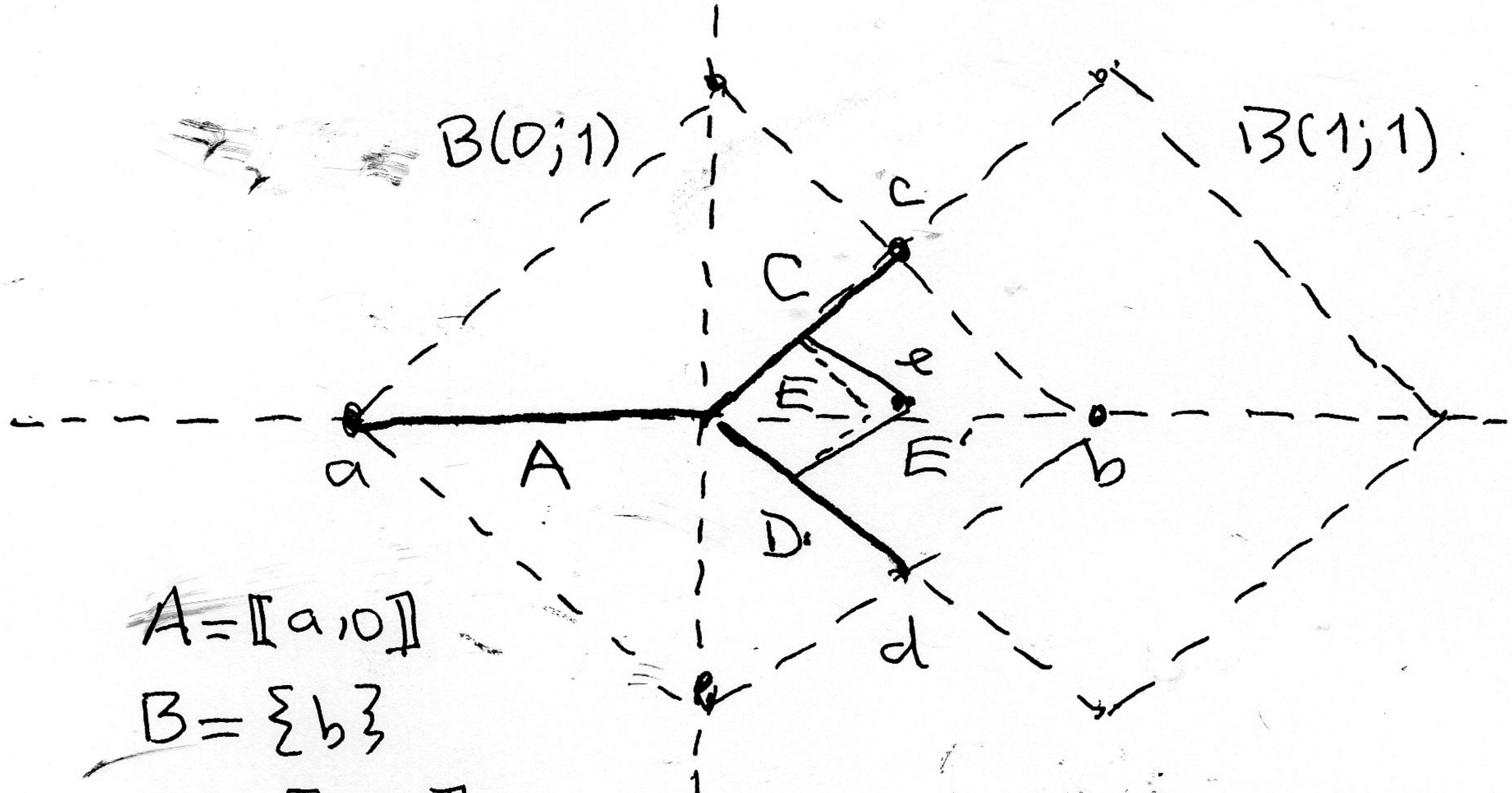
On the other hand, if we restrict our attention to $KL(X)$, then, by the Rådström extension theorem, $KL(X)$ embeds as a cone in a normed vector space whose metric restriction to $KL(X)$ is the Hausdorff metric. (The process resembles how one obtains the integers from the natural numbers, via equivalence classes of "differences.")

Consequently, a nondegenerate linear hyperinterval $[\![A, B]\!]$ in $KL(X)$ is isometric–via its parameterization–to a line segment. Moreover, we always have

$$[\![A, B]\!] \subseteq I_{KL(X)}(A, B).$$

From Corollary 7, a strictly convex normed vector space X has the property that all metric intervals in $KL(X)$ are metrically convex, as long as their brackets are singletons. However, strict convexity goes only so far.

8 Proposition. *Let X be a normed vector space of dimension at least two. Then there are $A, B \in KL(X)$, where A is a line segment and B is a singleton, such that $I_{KL(X)}(A, B)$ is not metrically convex.*



$A = \llbracket a, 0 \rrbracket$

$B = \{b\}$

$C = \llbracket 0, c \rrbracket$

$D = \llbracket 0, d \rrbracket$

$E = \frac{1}{2}C + \frac{1}{2}D$

$E' = \mathrm{HULL}\,(E \cup \{e'\})$

THANK YOU!