Betweenness and the Diameter Metric

(Inspired by recent joint work with Aisling McCluskey and Stephen Watson)

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1. Metric Betweenness Structures

The overarching theme of this study is the investigation of how notions of betweenness and distance interact, and so the fundamental objects we consider are *metric betweenness structures*.

These are triples $\langle X, I, \varrho \rangle$, where:

(1) $I = I(\cdot, \cdot, \cdot)$ is a ternary relation on underlying set X, where we read $I(a, c, b)$ as "point c lies between points a and b"; and

(2) $\varrho : X^2 \to [0, \infty)$ is a binary function, where we read $\varrho(a, b) = t$ as "points a and b are distance t from each other."

The distance function ϱ satisfies the classical axioms for real-valued metrics; the ternary relation I satisfies the following *basic* axioms:

- (Inclusivity) $I(a,a,b) \wedge I(a,b,b)$
- (Symmetry) $I(a,c,b) \rightarrow I(b,c,a)$
- (Uniqueness) $I(a,b,a) \rightarrow b = a$.

The pair $\langle X, I \rangle$ is a *betweenness structure* if the ternary relation I satisfies these three axioms.

“Just about” all betweenness structures considered here satisfy:

- (Transitivity) $(I(a,c,b) \wedge I(a,d,c)) \rightarrow I(a,d,b)$.

If we fix the first argument and rewrite $I(a,x,y)$ as $x \leq_a y$, then the transitivity axiom looks like usual binary transitivity for each relation $\leq_a$. When transitity is combined with inclusivity and uniqueness, each $\leq_a$ specifies a pre-ordering on X with unique least element a.

In addition there are axioms that hold in some interesting cases, but not in others.

- (Antisymmetry) $(I(a,c,b) \wedge I(a,b,c)) \to b = c$

- (Convexity) $(I(a,c,b) \wedge I(a,d,b) \wedge I(c,e,d)) \to I(a,e,b)$ (souped-up transitivity)

- (Disjunctivity) $I(a,d,b) \to (I(a,d,c) \vee I(b,d,c))$.

For betweenness structures satisfying both transitivity and antisymmetry, each binary relation $\leq_a$ specifies a partial ordering with unique minimal element a.

The set of points I-between a and b is denoted

$$I(a,b) := \{x \in X; I(a,x,b) \text{ holds}\}.$$

This is referred to as the *I-interval with bracket set* $\{a,b\}$.

(For example, in interval terms, the uniqueness axiom says $I(a,a) = \{a\}$, and transitivity says $I(a,c) \subseteq I(a,b)$ whenever $c \in I(a,b)$.)

Intervals may have several bracket sets; however if anti-symmetry holds for I, then two distinct bracket sets for the same interval must be disjoint. (For if $I(a,b) = I(a,c)$ then we have both $c \in I(a,b)$ and $b \in I(a,c)$; so $b = c$.)

1.1 Example. A *continuum* is a nonempty, connected, compact Hausdorff topological space. Given a continuum X, let $\mathcal{K}(X)$ be its family of *subcontinua*; i.e., subsets that are continua relative to their inherited topologies. For $a, b \in X$, the corresponding *subcontinuum interval* is

$$I_{\mathcal{K}}(a, b) := \bigcap\{A \in \mathcal{K}(X) : \{a, b\} \subseteq A\}.$$

The betweenness structure $\langle X, I \rangle$ (where $I = I_{\mathcal{K}}$) satisfies all the axioms above, with the exception of antisymmetry, which fails for the $\sin\frac{1}{x}$-continuum because each of two points on the spine lies subcontinuum-between the other point and a third point on the wavy bit.

Disjunctivity, for example, holds because the union of two overlapping subcontinua is a subcontinuum.

(Of particular importance in this talk are metric betweenness structures where X is a continuum, $I = I_{\mathcal{K}}$, and ϱ generates the continuum topology on X.)

We consider two natural ways in which a metric ϱ gives rise to betweenness: The first was studied by K. Menger in 1929:

$$I_\varrho^{+}(a, c, b) \text{ iff } \varrho(a, b) = \varrho(a, c) + \varrho(c, b);$$

the second seems to be new:

$$I_\varrho^{\sqcup}(a, c, b) \text{ iff } \varrho(a, b) = \varrho(a, c) \sqcup \varrho(c, b).$$

(where "$x \sqcup y$" is the infix version of "max$\{x, y\}$")

Both I_ϱ^+ and $I_\varrho^\sqcup$ satisfy the three basic axioms above, and Menger originally showed that I_ϱ^+ satisfies transitivity, as well as antisymmetry. (Both convexity and disjunctivity may fail for I_ϱ^+.)

For general metrics $I_\varrho^\sqcup$ does not satisfy even transitivity. HOWEVER: If ϱ is an *ultrametric*; i.e., if it satisfies the *strong triangle inequality*

$$\varrho(a,b) \leq \varrho(a,c) \sqcup \varrho(c,b),$$

then convexity and disjunctivity both hold for $I_\varrho^\sqcup$.

In fact, any two $I_\varrho^\sqcup$-intervals are either disjoint or nested.

If ϱ is an ultrametric on X, then $I_\varrho^\sqcup$ satisfies all the axioms above, except antisymmetry.

(Indeed, if $a, b, c \in X$ are distinct and $c \notin I_\varrho^\sqcup(a, b)$, then $I_\varrho^\sqcup(a, b, c)$ and $I_\varrho^\sqcup(b, a, c)$ both hold.)

Another axiom satisfied by $I_\varrho^\sqcup$—when ϱ is an ultrametric—is

- (Totality) $I(a,c,b) \vee I(a,b,c)$.

To see this, suppose $a,b,c \in X$. If $c \notin I(a,b)$, then $\varrho(a,b) < \varrho(a,c) = \varrho(b,c)$; i.e., triangles are "tall isosceles." Then $\varrho(a,b) \sqcup \varrho(b,c) = \varrho(b,c) = \varrho(a,c)$; hence $b \in I(a,c)$.

In the presence of transitivity, totality is a strengthening of disjunctivity.

Also totality, when interpreted as a property of the binary relations $\leq_a$, is the condition that makes pre-orderings into total ones. (As it turns out, however, having *each* $\leq_a$ be a total order—when antisymmetry holds—implies that the structure has at most two points.)

For $\langle X, I, \varrho \rangle$ a metric betweenness structure, we may relate I and ϱ as follows.

I is $+$-*induced* (resp., $+$-*subinduced* by ϱ if $I = I_{\varrho}^{+}$ (resp., $I \subseteq I_{\varrho}^{+}$).

Similarly we may define being $\sqcup$-*(sub)induced* when ϱ is an ultrametric.

Note that if I is $+$-subinduced by a metric, then it satisfies antisymmetry, because the conclusion of that axiom involves just equality.

For $r > 0$, we say I is *r-compatible* with ϱ if for all $a, b, c \in X$ with $I(a, c, b)$ holding, we have $\varrho(a, c) \leq r\varrho(a, b)$.

(This includes $\varrho(a, b) \leq r\varrho(a, b)$, which forces $r \geq 1$ in any nondegenerate structure.)

I is *compatible* with ϱ if it is r-compatible with ϱ for some $r > 0$.

Clearly if I is either $+$-subinduced or $\sqcup$-subinduced by ϱ, then I is 1-compatible with ϱ.

Observe that if I is compatible with ϱ then all I-intervals are ϱ-bounded: for if $x, y \in I(a, b)$ and I is r-compatible with ϱ, then $\varrho(x, y) \leq \varrho(a, x) + \varrho(a, y) \leq 2r\varrho(a, b)$.

2. Diameter Distance

If $\langle X, I, \varrho \rangle$ is a metric betweenness structure and $A \subseteq X$, define the ϱ-*diameter* of A to be

$$\Delta(A) = \Delta_\varrho(A) := \text{sup}\{\varrho(x, y) : x, y \in A\}.$$

We then define the *diameter distance* associated with ϱ by

$$\varrho^*(a, b) := \Delta(I(a, b)).$$

If I is r-compatible with ϱ, then—from above—we have

$$\varrho(a, b) \leq \varrho^*(a, b) \leq 2r\varrho(a, b).$$

2.1 Proposition. Let $\langle X, I, \varrho \rangle$ be a metric betweenness structure.

(i) If I is $+$-subinduced by ϱ, then $\varrho^* = \varrho$.

(ii) If ϱ is an ultrametric, then I is $\sqcup$-subinduced by ϱ if and only if $\varrho^* = \varrho$.

Proof. Fix $a, b \in X$. Then for any $x, y \in X$ we have both

(1) $\varrho(x, y) \leq \varrho(a, x) + \varrho(a, y)$ and

(2) $\varrho(x, y) \leq \varrho(x, b) + \varrho(y, b)$;

and when we add these two inequalities we get

(3) $2\varrho(x, y) \leq (\varrho(a, x) + \varrho(x, b)) + (\varrho(a, y) + \varrho(y, b))$.

If I is $+$-subinduced by ϱ and $x, y \in I(a, b)$, then the right-hand side of (3) equals $2\varrho(a, b)$; hence $\varrho^*(a, b) \leq \varrho(a, b)$. Thus we have equality.

If now ϱ is an ultrametric, we take the join of the strong triangle inequality versions of (1) and (2) to obtain

(4) $\varrho(x, y) \leq (\varrho(a, x) \sqcup \varrho(x, b)) \sqcup (\varrho(a, y) \sqcup \varrho(y, b))$.

If I is $\sqcup$-subinduced by ϱ and $x, y \in I(a, b)$, then the right-hand side of (4) equals $\varrho(a, b)$; and again $\varrho^*(a, b) \leq \varrho(a, b)$. This establishes the *only if* parts of (i) and (ii).

To prove the *if* half of (ii), suppose ϱ is an ultrametric and that $a, b, c \in X$ are such that $c \in I(a,b) \setminus I_\varrho^\sqcup(a,b)$. Then–because ϱ is an *ultra*metric–we have $\varrho(a,b) < \varrho(a,c) = \varrho(b,c)$. But then–because $I(a,c,b)$ holds–we have $\varrho^*(a,b) \geq \varrho(a,c) > \varrho(a,b)$; so $\varrho^* \neq \varrho$. $\square$

There is no converse to Proposition 2.1 (i) above.

2.2 Example. A metric betwenness structure $\langle X, I, \varrho \rangle$ such that $\varrho^* = \varrho$ but it is not the case that $I \subseteq I_\varrho^+$:

Set $X = [0,1] \subseteq \mathbb{R}$, where $I = I_\mathcal{K}$ (so the betweenness intervals are the usual closed intervals of $[0,1]$) and $\varrho(x,y) := \sqrt{|x-y|}$. Then I_ϱ^+ is *trivial*–i.e., no interval contains more than two points– while I is not; so $I \not\subseteq I_\varrho^+$. On the other hand, $\varrho^* = \varrho$.

Some terminology: (1) If ϱ and σ are two metrics on X, σ *refines* ϱ if the topology generated by σ is finer than that generated by ϱ. If each metric refines the other, then they're termed *equivalent*.

(2) If there is a real $t > 0$ such that $\varrho(a, b) \leq t\sigma(a, b)$ for all $a, b \in X$, then σ *strongly refines* ϱ. If each metric strongly refines the other, then they're termed *strongly equivalent*.

Strong refinement straightforwardly implies refinement; the converse is not true, even if the underlying space is compact: For $X = [0, 1]$, the closed unit interval, let $\varrho(x, y) := |x - y|$, and let $\sigma(x, y) = \sqrt{\varrho(x, y)}$. Then ϱ and σ are equivalent and σ strongly refines ϱ, but ϱ does not strongly refine σ.

2.3 Proposition. Let $\langle X, I, \varrho\rangle$ be a metric betweenness structure such that I satisfies transitivity and disjunctivity and ϱ is I-bounded (e.g., $I = I_{\mathcal{K}}$, where X is a continuum, and ϱ generates the continuum topology). Then:

(i) The diameter distance ϱ^* is a metric on X that is 1-compatible with I and strongly refines ϱ.

(ii) I and ϱ are compatible if and only if ϱ^* is strongly equivalent to ϱ.

(iii) ϱ^{**} is a metric on X that is 1-compatible with I and strongly equivalent to ϱ^*.

(iv) If I satisfies totality, then ϱ^* is an ultrametric on X that $\sqcup$-subinduces I and strongly refines ϱ. Moreover, $\varrho^{**} = \varrho^*$.

Proof. Ad (i). Clearly $\varrho \leq \varrho^*$; so if ϱ^* is a metric then it strongly refines ϱ.

Symmetry and positive-definiteness are obvious; we show that the triangle inequality holds. Indeed, suppose $a, b, c \in X$. We use the well-known fact that if sets A and B overlap, then $\Delta(A \cup B) \leq \Delta(A) + \Delta(B)$. We also use the assumption of disjunctivity:

$$\varrho^*(a, b) = \Delta(I(a, b)) \leq \Delta(I(a, c) \cup I(c, b)) \leq$$

$$\leq \Delta(I(a, c)) + \Delta(I(c, b)) = \varrho^*(a, c) + \varrho^*(c, b).$$

As for 1-compatibility, use transitivity: If $c \in I(a, b)$ then $I(a, c) \subseteq I(a, b)$. Hence

$$\varrho^*(a, c) = \Delta(I(a, c)) \leq$$

$$\leq \Delta(I(a, b)) = \varrho^*(a, b).$$

Ad (ii). Assume I and ϱ are r-compatible, $r > 0$, and fix $a, b \in X$. For $x, y \in I(a, b)$ arbitrary, we have

$$\varrho(x, y) \leq \varrho(a, x) + \varrho(a, y) \leq 2r\varrho(a, b).$$

Hence $\varrho^* \leq 2r\varrho$, showing ϱ strongly refines ϱ^*. By (i), this makes the two metrics strongly equivalent.

For the converse, assume $\varrho^* \leq r\varrho$ for some $r > 0$. For $a, b, c \in X$ such that $c \in I(a, b)$, we have

$$\varrho(a, c) \leq \varrho^*(a, b) \leq r\varrho(a, b);$$

hence I and ϱ are r-compatible.

Ad (iii). This follows from (i) and (ii), plus the fact that compatible implies I-bounded.

Ad (iv). If I satisfies totality, then $\{I(a,c), I(c,b)\}$ is nested; hence either $I(a,b) \subseteq I(a,c)$ or $I(a,b) \subseteq I(c,b)$. Thus

$$\varrho^*(a,b) \leq \varrho^*(a,c) \sqcup \varrho^*(c,b),$$

so ϱ^* is an ultrametric.

As for $\sqcup$-subinducement, suppose $I(a,c,b)$ holds. Then $I(a,b) = I(a,c)$ or $I(a,b) = I(c,b)$, by totality. Hence

$$\varrho^*(a,b) = \varrho^*(a,c) \sqcup \varrho^*(c,b),$$

and so $I \subseteq I^{\sqcup}_{\varrho^*}$. That $\varrho^{**} = \varrho^*$ immediately follows from Proposition 2.1 (ii). $\Box$

3. Metric Continua

A *metric continuum* is a pair $\langle X, \varrho \rangle$, where X is a continuum and ϱ is a generating metric.

The metric continuum $\langle X, \varrho \rangle$ is viewed here as a metric betweenness structure, where $I = I_{\mathcal{K}}$.

A continuum X is *decomposable* if it is the union of two proper subcontinua; *indecomposable* otherwise. Being *hereditarily* indecomposable means that all subcontinua are indecomposable, and this is equivalent to saying that $\mathcal{K}(X)$ is a *rank one family*; i.e., any two subcontinua are either disjoint or nested.

And this means that the subcontinuum betweenness of X satisfies totality.

3.1 Corollary. Let $\langle X, \varrho \rangle$ be a hereditarily indecomposable metric continuum. Then ϱ^* is an ultrametric on X that $\sqcup$-subinduces I and strongly refines ϱ. Moreover, $\varrho^{**} = \varrho^*$ and ϱ is not compatible with I.

Proof. Because I satisfies totality, we may invoke Proposition 2.3 (iv) to establish all but the assertion that I and ϱ are incompatible. But by Proposition 2.3 (ii), compatibility would imply that ϱ^* and ϱ are strongly equivalent. This is impossible because ultrametrics are well known to generate zero-dimensional topologies. $\square$

Define a continuum X to be *diameter stable* if whenever ϱ is a generating metric, the ϱ^*-topology is equal to the ϱ-topology. The continuum is *diameter unstable* if each ϱ^*-topology properly contains the ϱ-topology (and is thus not homeomorphic to it). Hence nondegenerate hereditarily indecomposable continua are diameter unstable.

Given a metrizable continuum X, there are at most two topologies on X that arise from the diameter process applied to any given generating metric. We know that two strongly equivalent generating metrics induce strongly equivalent diameter metrics, but we do not have an answer for the following.

3.2 Question. If X is a continuum with two generating metrics ϱ and σ, is it true that ϱ^* and σ^* are equivalent?

[Post-talk comment: The answer to this question is YES.]

If $\langle X, \varrho \rangle$ is a nondegenerate metric continuum, then ϱ^* can never have isolated points. In fact we have the following stronger result–very much like boundary bumping–which shows each nonempty ϱ^*-open set to have $\mathfrak{c} := 2^{\aleph_0}$ points.

3.3 Proposition Let $\langle X, \varrho \rangle$ be a nondegenerate metric continuum, with $a \in X$ and $U \subseteq X$ a ϱ^*-open set containing a. Then there is a nondegenerate $K \in \mathcal{K}(X)$ with $a \in K \subseteq U$.

Proof. Fix $a \in U \subseteq X$. Then for some $r > 0$ the ϱ^*-ball neighborhood $B_{\varrho^*}(a; r)$ is contained in U. Using ordinary boundary bumping for $\langle X, \varrho \rangle$, fix nondegenerate $K \in \mathcal{K}(X)$ such that

$$a \in K \subseteq B_{\varrho}(a; \tfrac{r}{3}).$$

Given any $b \in K$ we have $I(a, b) \subseteq K$. So pick any $x, y \in I(a, b)$. Then

$$\varrho(x, y) \leq \varrho(a, x) + \varrho(a, y) < \tfrac{2r}{3};$$

so

$$\varrho^*(a, b) = \Delta_{\varrho}(I(a, b)) \leq \tfrac{2r}{3} < r.$$

Hence $b \in B_{\varrho^*}(a; r)$. Since $b \in K$ is arbitrary, we have

$$a \in K \subseteq B_{\varrho^*}(a; r) \subseteq U,$$

as desired. $\square$

If X is a continuum, with $a \in X$, then the *composant* of X at a is the union of all proper subcontinua of X that contain a. Every decomposable continuum has either one or three separate composants; however if a metrizable continuum is nondegenerate and indecomposable, it has precisely $\mathfrak{c}$ pairwise disjoint composants.

3.4 Proposition Let $\langle X, \varrho \rangle$ be a metric continuum. Then each of its composants is ϱ^*-open. Hence, if X is nondegenerate and indecomposable, then it can be partitioned into $\mathfrak{c}$ clopen sets. In particular, I and ϱ are incompatible (and the metric space $\langle X, \varrho^* \rangle$ has weight $\mathfrak{c}$).

Proof. If X is degenerate there's nothing to prove. Otherwise, let C be any composant of X, with $a \in C$. Let $0 < r \leq \Delta_\varrho(X)$. We show that $B_{\varrho^*}(a; r) \subseteq C$. Indeed, if $x \in B_{\varrho^*}(a; r)$, then $r > \varrho^*(a, x) = \Delta_\varrho(I(a, x))$, implying that $I(a, x) \neq X$. Hence not every subcontinuum of X containing $\{a, x\}$ is all of X, so let $A \in \mathcal{K}(X) \setminus \{X\}$ contain $\{a.x\}$. Thus $x \in A \subseteq C$, and we conclude that C is ϱ^*-open.

If X is nondegenerate and indecomposable, then all $\mathfrak{c}$ of its pairwise disjoint composants are open, and hence closed as well. Hence the ϱ^*-topology is quite distinct from the ϱ-topology, implying that I and ϱ are incompatible, by Proposition 2.3 (ii). $\square$

By Proposition 3.4, all nondegenerate indecomposable continua are diameter unstable, not just the hereditarily indecomposable ones.

But decomposable continua can be diameter unstable as well.

3.5 Example Let $X = A \cup S$ be the $\sin \frac{1}{x}$-continuum in the plane $\mathbb{R}^2$, where $A = \{0\} \times [-1, 1]$ and $S = \{\langle x, \sin \frac{1}{x} \rangle : 0 < x \leq 1\}$. With ϱ the inherited Euclidian metric, $\langle X, \varrho^* \rangle$ is homeomorphic to $X' = A' \cup S \subseteq \mathbb{R}^2$, where $A' = \{-1\} \times [-1, 1]$. In particular, ϱ^* generates a disconnected topology. This works for any metric equivalent to ϱ: the reason is that A is a nondegenerate proper subcontinuum which is *terminal*, in the sense that any subcontinuum of X hitting both A and $X \setminus A$ must contain A. Such subcontinua are invariably clopen relative to any diameter metric, and this shows that X is a diameter unstable continuum.

We saw that if $\langle X, \varrho \rangle$ is a hereditatily indecomposable metric continuum, then I is $\sqcup$-subinduced by ϱ^*.

3.6 Question. In the situation above, is it possible for ϱ^* to actually $\sqcup$-*induce* I?

We can also ask the question of whether I can be $+$-(sub)induced by ϱ^*. To this end, we have a trivial answer and a less trivial one: The trivial answer is to let $\langle X, \varrho \rangle$ be any metric continuum where I is trivial (a simple closed curve, say). Then $\varrho^* = \varrho$. And by letting σ be $\sqrt{\varrho}$, we have that $\sigma^* = \sigma$, both I and I_σ^+ are trivial, and therefore $I = I_{\sigma^*}^+$. In particular, every continuum with trivial subcontinuum betweenness is diameter stable.

Another, less trivial, source of diameter stability uses *convex* metrics.

By classical work of Menger, Bing and Moise, a metrizable continuum X has a convex metric ϱ–i.e., where each non-degenerate I_ϱ^+-interval has more than two points–if and only if X is locally connected.

Given such a metric continuum $\langle X, \varrho \rangle$ and $a, b \in X$ distinct, then, there is an isometry $f : [0, \varrho(a,b)] \to X$ with $f(0) = a$ and $f(\varrho(a,b)) = b$. If $I(a,c,b)$ holds, then $c = f(t)$ for some $0 \leq t \leq \varrho(a,b)$; hence $\varrho(a,b) = t + (\varrho(a,b) - t) = \varrho(a,c) + \varrho(c,b)$, and $I \subseteq I_\varrho^+$.

If A is the arc that is the image of the isometry f, then $\varrho^*(a,b) \leq \Delta_\varrho(A) = \varrho(a,b)$; so $\varrho^* = \varrho$.

In the event X is also hereditarily unicoherent, then $A = I(a,b)$; hence $I = I_\varrho^+ = I_{\varrho^*}^+$. [Post-talk comment: Because of the affirmative answer to Question 3.2, we know that locally connected metrizable continua are diameter stable.]

THANK YOU