Straightenable Continua and the Stretch Topology

(Joint with Daron Anderson and Aisling McCluskey)

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Galway Topology Colloauium, 09 July, 2026

1. Straightening metrics

A *continuum* is a connected compact Hausdorff space; $\mathcal{K}(X)$ denotes the family of subcontinua of continuum X. If $a, b, c \in X$, we say c is *subcontinuum-between* a and b if every subcontinuum containing $\{a, b\}$ also contains c.

We denote by $[a, b]$ the set $\bigcap\{K \in \mathcal{K}(X) : a, b \in K\}$, the *subcontinuum interval with bracket points* a, b. (Always compact, but connected iff X is *hereditarily unicoherent*.)

A metric ϱ on continuum $\langle X, \mathcal{T}\rangle$ is *generating* if its metric topology $\mathcal{T}_\varrho$ coincides with the continuum topology $\mathcal{T}$. ϱ is *straightening* if $\varrho(c, d) \leq \varrho(a, b)$ whenever $c, d \in [a, b]$.

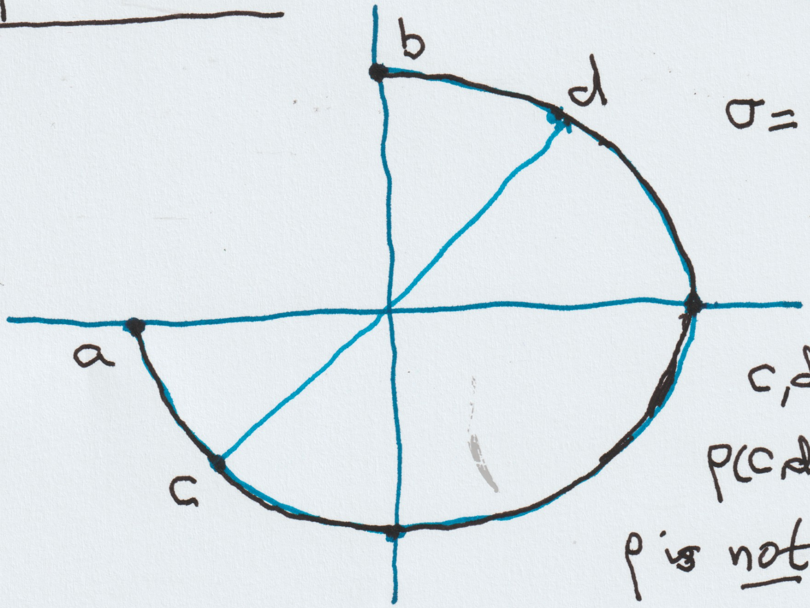
The continuum is *straightenable* if it has a generating metric that is also straightening.

Example 1.

$\mathcal{X}$ = wraparound arc

ρ = euclidean metric

σ = Arclength metric



$c, d \in [a, b]$

$\rho(c,d) > \rho(a,b)$

ρ is not straightening

BUT $\sigma \equiv \rho$ is straightening

So metrizable arcs are straightenable. An example of a continuum that is *not* straightenable is a Knaster buckethandle- or any nondegenerate indecomposable metrizable continuum X. (To see this, let a and b lie in separate composants of X. Then $[a, b] = X$, while a and b may be arbitrarily close relative to any generating metric.)

Straightenability is clearly a topological invariant: if $h : X \to Y$ is a homeomorphism between continua, then h preserves subcontinuum intervals. Hence if ϱ is a straightening metric on X, then ϱ^h, the induced metric on Y defined by $\varrho^h(a, b) := \varrho(h^{-1}(a), h^{-1}(b))$, is itself a straightening metric on Y.

2. Uniformizing metrics

Let X be a continuum, with ϱ a metric on (the underlying domain of) X. ϱ is *uniformizing* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $\varrho(a,b) \leq \delta$, we have $\varrho(c,d) \leq \varepsilon$ for all $c, d \in [a,b]$.

Clearly straightening metrics are uniformizing; just let δ equal ε.

In the wraparound arc (Example 1) above, ϱ is uniformizing without being straightening. (Indeed, $\varrho(x,y) \leq \sqrt{2}\varrho(u,v)$ whenever $x, y \in [u,v]$.)

No generating metric for a nondegenerate indecomposable continuum is uniformizing. (Just pick ε smaller than the continuum's diameter.)

3. Stretch metrics and stretch topologies

Let X be a continuum. A metric ϱ on X is *interval-bounded* if $\mathrm{diam}_\varrho([a,b]) < \infty$ for all $a, b \in X$. If ϱ is an interval-bounded metric on X and $a, b \in X$ we define

$$\varrho^*(a,b) := \mathrm{diam}_\varrho([a,b]).$$

Proposition 1. If ϱ is an interval-bounded metric on continuum X, then ϱ^* is a metric.

Proof. The only nontrivial condition to check is the triangle inequality. So let $a, b, c \in X$ be given. Then the *disjunctivity* condition holds for subcontinuum betweenness; i.e., we have $[a,b] \subseteq [a,c] \cup [c,b]$. Also we use the well-known fact that the diameter of the union of two overlapping sets is bounded by the sum of the separate diameters. Hence we have $\varrho^*(a,b) = \mathrm{diam}_\varrho([a,b]) \leq \mathrm{diam}_\varrho([a,c] \cup [c,b]) \leq \mathrm{diam}_\varrho([a,c]) + \mathrm{diam}_\varrho([c,b]) = \varrho^*(a,c) + \varrho^*(c,b)$. $\square$

If ϱ is an interval-bounded (e.g., generating) metric on a continuum X, we clearly have the inequality $\varrho \leq \varrho^*$ and therefore refer to ϱ^* as the *stretch* of ϱ. Stretching is "idempotent," in the following sense.

Proposition 2. If ϱ is an interval-bounded metric on continuum X, then ϱ^* is also interval-bounded, and $\varrho^{**} := (\varrho^*)^* = \varrho^*$.

Proof. Let $a, b \in X$ be given. Then $\varrho^{**}(a, b) = \text{diam}_{\varrho^*}([a, b])$, by definition. Hence it is enough to show that the right-hand side is equal to $\text{diam}_{\varrho}([a, b])$. Indeed, if $c, d \in [a, b]$ are arbitrary, then $\varrho^*(c, d) = \text{diam}_{\varrho}([c, d])$. But subcontinuum betweenness is *convex*, in the sense that $[c, d] \subseteq [a, b]$ whenever $c, d \in [a, b]$. Hence $\varrho^*(c, d) \leq \text{diam}_{\varrho}([a, b]) = \varrho^*(a, b)$. By taking the supremum over all $c, d \in [a, b]$, we see that $\varrho^{**}(a, b) \leq \varrho^*(a, b)$. $\square$

Corollary 1. The stretch of any interval-bounded metric (e.g., a generating metric) on a continuum is a straightening metric.

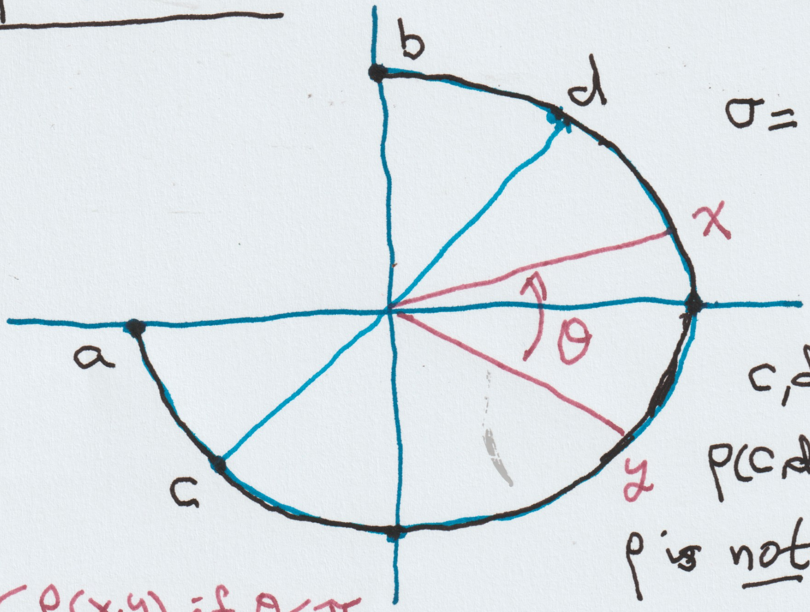
Let's calculate ϱ^* in the wraparound arc (Example 1) above.

Example 2.

X = wraparound arc

$\rho \neq$ euclidean metric

$\sigma =$ Arclength metric



$c, d \in [a,b]$

$\rho(c,d) > \rho(a,b)$

ρ is <u>not</u> straightening

<u>BUT</u> $\sigma \equiv \rho$ is straightening

$$\rho^*(x,y) = \begin{cases} \rho(x,y) \text{ if } \theta \leq \pi \\ \rho(c,d) = 2 \text{ if } \theta \geq \pi \end{cases}$$

Some terminology: (1) If ϱ and σ are two metrics on X, σ *topologically refines* ϱ if the topology $\mathcal{T}_\sigma$ generated by σ is finer than that generated by ϱ. (I.e., $\mathcal{T}_\sigma \supseteq \mathcal{T}_\varrho$.) If each metric refines the other, then they're termed *topologically equivalent*.

(2) σ *uniformly refines* ϱ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\sigma(a,b) \leq \delta$ then $\varrho(a,b) \leq \varepsilon$. If each metric uniformly refines the other, then they're termed *uniformly equivalent*.

Uniform refinement/equivalence implies topological refinement/equivalence; the following is well known.

Lemma 1. If two compact metrics are topologically equivalent then they're uniformly equivalent.

Proposition 3. Assume ϱ is an interval-bounded metric on a continuum X. The following conditions are equivalent.
(i) The stretch metric ϱ^* is uniformly equivalent to ϱ.
(ii) ϱ uniformly refines ϱ^*.
(iii) ϱ^* is uniformizing.

Proof. ϱ^* is a metric, thanks to Proposition 1. ϱ^* uniformly refines ϱ automatically, by virtue of the inequality $\varrho^* \geq \varrho$, so the first "iff" is immediate. The second "iff" is just a reworking of the relevant definitions. $\square$

In a continuum X, a metric ϱ is *interval-compact* if each subcontinuum interval is compact in the topology generated by ϱ.

Proposition 4. Let ϱ and σ be two metrics on continuum X. If σ is interval-compact and topologically (resp., uniformly) refines ϱ, then ϱ is interval-compact and σ^* topologically (resp., uniformly) refines ϱ^*.

Proof. We show the case where refinement is topological; the uniform case is handled similarly.

Assuming σ is interval-compact and topologically refines ϱ, we know immediately that ϱ is interval-compact, because compactness is stable under coarsification. Hence both metrics are interval-bounded, implying that both ϱ^* and σ^* are metrics (by Proposition 1). It remains to show $\mathcal{T}_{\sigma^*} \supseteq \mathcal{T}_{\varrho^*}$.

For $a \in X$ and $r > 0$, $B_\varrho(a;r) := \{x \in X : \varrho(a,x) < r\}$ is the *open ϱ-ball of radius r, centered at a*. So to prove that σ^* topologically refines ϱ^*, it suffices to fix $a \in X$ and $r > 0$, and find $s > 0$ such that $B_{\sigma^*}(a;s) \subseteq B_{\varrho^*}(a;r)$.

Since σ topologically refines ϱ, we choose $s > 0$ such that $B_\sigma(a;s) \subseteq B_\varrho(a;\frac{r}{2})$.

Suppose $b \in X$ is such that $\sigma^*(a,b) < s$. We need to show that $\varrho^*(a,b) < r$.

For any $x, y \in [a,b]$, we have $\sigma(a,x) \leq \sigma^*(a,b) < s$ and $\sigma(a,y) \leq \sigma^*(a,b) < s$; hence both $\varrho(a,x)$ and $\varrho(a,y)$ are $< \frac{r}{2}$. By the fact that ϱ is interval-compact, we may choose $x, y \in [a,b]$ such that $\varrho^*(a,b) = \varrho(x,y)$.

Hence $\varrho^*(a,b) = \varrho(x,y) \leq \varrho(a,x) + \varrho(a,y) < \frac{r}{2} + \frac{r}{2} = r$. $\Box$

The following is an immediate consequence of Lemma 1, Proposition 4, and the fact that a sequence is Cauchy relative to one metric if it's Cauchy relative to a uniformly equivalent metric.

Corollary 2. Let X be a continuum, with ϱ and σ two of its generating metrics. Then the respective stretch metrics are uniformly equivalent. (In particular, the respective stretch metrics are either both complete or both incomplete.)

This result allows us to make the following definition.

Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum. Then its *stretch topology*, denoted $\mathcal{T}^*$, is any one of the (equal) metric topologies $\mathcal{T}_{\varrho^*}$, where ϱ is a generating metric for $\mathcal{T}$.

The stretch topology always refines the continuum topology; hence the two topologies coincide iff the stretch topology is compact.

Corollary 3. Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum; the following conditions are eqivalent.
(i) $\langle X, \mathcal{T} \rangle$ is straightenable.
(ii) $\mathcal{T}^* = \mathcal{T}$.
(iii) $\mathcal{T}^*$ is a compact topology.
(iv) $\langle X, \mathcal{T} \rangle$ has a generating metric that is uniformizing.
(v) Every generating metric for $\langle X, \mathcal{T} \rangle$ is uniformizing.

"Straightenable metrizable continua are the ones where you can stretch the metric in order to straighten the subcontinuum intervals, but still not break the topology."

Remark 1. Having a generating metric that is straightening is a “geometric” property of a continuum, which is also a topological invariant. Another–very famous–example of this is having a generating metric that is *convex*. (This is a generating metric ϱ with the property that for any two distinct points a, b, there is a third point c such that $\varrho(a, b) = \varrho(a, c) + \varrho(c, b)$. I.e., every two points have a third point *metrically between* them.) This notion was introduced by K. Menger in 1929; there he showed that any continuum possessing a convex metric must be locally connected. Almost twenty years after that, R.H. Bing and E. Moise independently showed that every locally connected continuum has a convex metric, thereby establishing that this “geometric” property of continua may be characterized in purely topological language.

What we lack at present is such a characterization of straightenability.

4. Continua that are not straightenable

We next consider *stretch-topological* properties of nonstraightenable continua.

For instance, they share with nondegenerate continuum topologies the lack of isolated points.

Proposition 5. Let X be a nondegenerate metrizable continuum, with $a \in X$ and $U \subseteq X$ a stretch-open set containing a. Then there is a nondegenerate $K \in \mathcal{K}(X)$ with $a \in K \subseteq U$.

Proof. Fix generating metric ϱ and $a \in U \subseteq X$, where U is ϱ^*-open. Then for some $r > 0$ we have the ϱ^*-open ball $B_{\varrho^*}(a; r)$ contained in U. Using ordinary boundary bumping for $\langle X, \varrho \rangle$, fix nondegenerate $K \in \mathcal{K}(X)$ such that

$a \in K \subseteq B_{\varrho}(a; \frac{r}{3})$.

Given any $b \in K$ we have $[a, b] \subseteq K$. So pick $x, y \in [a, b]$ arbitrary. Then

$\varrho(x, y) \leq \varrho(a, x) + \varrho(a, y) < \frac{2r}{3}$;

so $\varrho^*(a, b) = \mathrm{diam}_{\varrho}([a, b]) \leq \frac{2r}{3} < r$.

Hence $b \in B_{\varrho^*}(a; r)$. Since $b \in K$ is arbitrary, we have

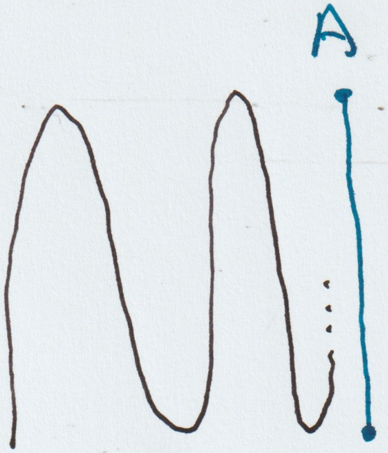
$a \in K \subseteq B_{\varrho^*}(a; r) \subseteq U$, as desired. $\Box$

Certain well-known types of subset of a continuum behave differently when viewed from the stretch-topology perspective.

Let $\langle X, \mathcal{T} \rangle$ be any continuum, with K a proper nondegenerate subcontinuum of X. We call K a *spine* (a.k.a. *terminal subcontinuum*) of X if whenever $L \in \mathcal{K}(X)$ intersects both K and $X \setminus K$, then $L \supseteq K$.

K is called a *semi-spine* of X if there is a point $v \in K$ (called a *vertex* of K) such that whenever $L \in \mathcal{K}(X)$ intersects both K and $X \setminus K$, then L contains v.

Example 3.



A is a spine

All maximal segments are semi-spines; only A has empty interior

Proposition 6. Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum.
(i) If K is a spine of X, then K is stretch-open; hence K is a proper stretch-clopen set and $\mathcal{T}^*$ is therefore a disconnected topology.
(ii) If K is a semi-spine of X, and $v \in K$ is a vertex of K, then every $a \in K \setminus \{v\}$ is in the stretch-interior of K.

We'll prove (i); (ii) is similar.

Proof. Fix generating metric ϱ, spine K, and $a \in K$. Since K is nondegenerate, fix $p \in K \setminus \{a\}$ and $r = \varrho(a,p)$. Then $B_{\varrho^*}(a;r) \subseteq K$: otherwise we have some $b \in X \setminus K$, with $\text{diam}_\varrho([a,b]) = \varrho^*(a,b) < r$. But then every subcontinuum containing $\{a,b\}$ contains K; hence $p \in [a,b]$, and $\text{diam}_\varrho([a,b]) \geq r$. $\Box$

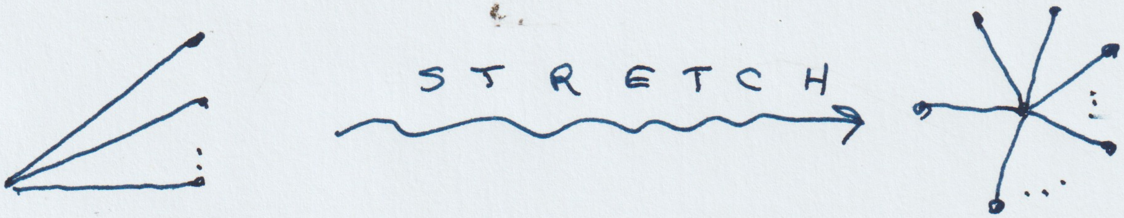
Corollary 4. Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum that contains either a spine or a nowhere dense semi-spine. Then X is non-straightenable. Moreover, if X has a spine, then $\langle X, \mathcal{T}^* \rangle$ fails to be connected.

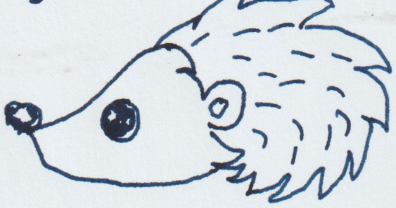
In Examples 4,5,6 below, we use the facts that: (1) the stretch topologies of subcontinua of a hereditarily unicoherent continuum are inherited from the stretch topology of the ambient continuum; and (2) arcs are straightenable (Example 1).

Example 4. The usual sin $\frac{1}{x}$-continuum is non-straightenable because its limiting arc is a spine. Under the stretch topology, this continuum becomes the disjoint union of an arc and a ray.

Example 5. By a *fan* we mean the cone over an infinite Boolean space. If a is a limit point of that subspace (of end points) and v is the apex of the cone, then the arc with end points v and a is a semi-spine (v is a vertex) with empty interior. Hence no fan is straightenable. Under the stretch topology, this continuum becomes a hedgehog space. The hedgehog metric induces subcontinuum betweenness in the original fan, while not being a generating metric.


STRETCH


STRETCH



If $\langle X, \mathcal{T} \rangle$ is a continuum and $a \in X$, the *composant of* X *at* a is the union of all proper subcontinua of X that contain a. Each composant is a connected dense subset.

A continuum is *decomposable* if it is the union of two proper subcontinua; *indecomposable* otherwise. In indecomposable nondegenerate metrizable continua, the composants form a partition into $\mathfrak{c} := 2^{\aleph_0}$ dense subsets. In particular, the composants of such a continuum are not open in the continuum topology.

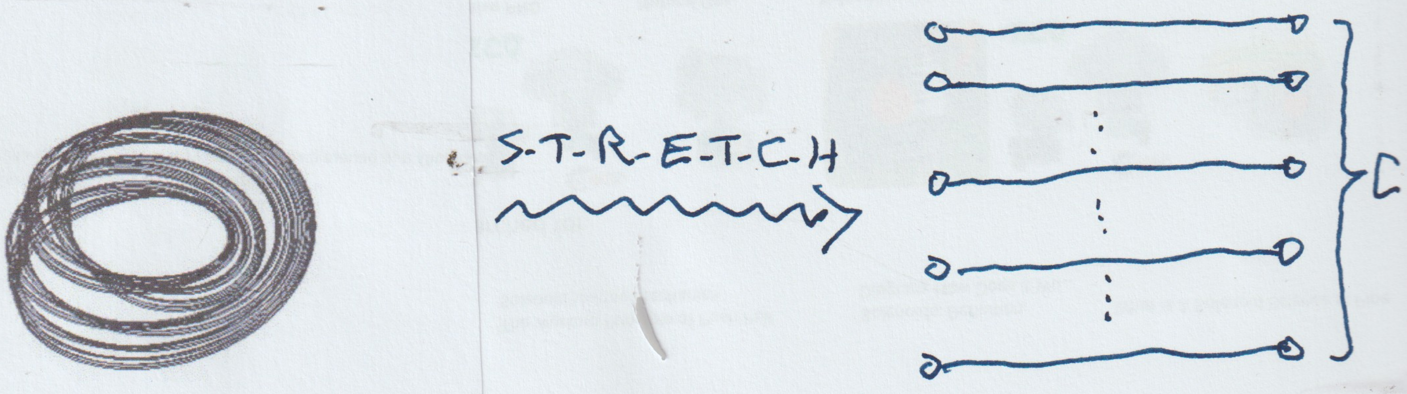
Proposition 7. Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum. Then every composant of X is stretch-open.

Proof. Given generating metric ϱ for $\mathcal{T}$, let $r = \text{diam}_{\varrho}(X)$. If $a, b \in X$ lie in separate composants of X then $\varrho(a, b) = r$. So given composant $C \subseteq X$ and $a \in C$, we have $B_{\varrho^*}(a; r) \subseteq C$. $\square$

From Proposition 7 and what we know about the composant structure of indecomposable metrizable continua, we have the following.

Corollary 5. If $\langle X, \mathcal{T} \rangle$ is a nondegenerate indecomposable continuum, then X is non-straightenable. Moreover, its family of composants relative to $\mathcal{T}$ becomes a decomposition of X into $\mathfrak{c}$-many stretch-clopen sets.

Example 6. The *dyadic solenoid* is the limit of an inverse ω-indexed system of unit circles in the complex plane, with squaring as the bonding map. Because it is nondegenerate and indecomposable, it is non-straightenable. Under the stretch topology, the composants become homeomorphic to the real line; hence the space is the free union (coproduct) of $\mathfrak{c}$-many copies of $\mathbb{R}$.


S-T-R-E-T-C-H
C

If X is a nondegenerate indecomposable metrizable continuum, and $a, b, c \in X$ are distinct and such that neither b nor c lies in the composant containing a, then $[a, b] = [a, c] = X$. In particular, subcontinuum betweenness for X fails to satisfy the *antisymmetry condition*: If c lies between a and b, and b lies between a and c, then $b = c$.

This condition was independently (and much earlier) discovered by B.E. Wilder, who called it Property C: given three distinct points a, b, c, there is a subcontinuum that contains a and exactly one of $\{b, c\}$. We refer to a continuum satisfying antisymmetry–or Wilder's Condition C–as a *Wilder continuum*

So indecomposable metrizable continua are clearly non-Wilder; the following then improves on Corollary 5.

Proposition 8. Every non-Wilder metrizable continuum is non-straightenable.

Proof. Assume $\langle X, \mathcal{T} \rangle$ is non-Wilder as well as straightenable. We derive a contradiction.

Being a non-Wilder continuum means that there are three distinct points $a, b, c \in X$ such that $[a, b] = [a, c]$. Let C be the union of all subcontinua of X that contain a and exclude b. Then, by assumption, we have $c \notin C$.

Let C' be the closure of C in X. If $b \notin C'$, then the subcontinuum C' is contained in the open set $X \setminus \{b\}$, and boundary bumping implies the existence of a subcontinuum $K \subseteq X \setminus \{b\}$, which properly contains C'. This contradicts the definition of C; so we conclude $b \in C'$.

Let ϱ be a generating metric for X, and fix a sequence $\langle b_n \rangle$ that consists of points of C and which ϱ-converges to b. By straightenability, we know that $\langle b_n \rangle$ ϱ^*-converges to b as well, implying that the diameters $\operatorname{diam}_\varrho([b_n, b])$ become arbitrarily small. Since $c \neq b$, there is an index m such that $c \notin [b_m, b]$. Hence there is a subcontinuum K that contains $\{b_m, b\}$ but does not contain c. Since $b_m \in C$, there is a subcontinuum $L \subseteq C$ that contains $\{a, b_m\}$. In particular, we know $c \notin L$. Thus $K \cup L$ is a subcontinuum containing $\{a, b\}$ but not c, and we conclude $c \notin [a, b]$, a contradiction. $\square$

5. Continua that are straightenable

By classical work of Menger, Bing and Moise (see Remark 1 above), a metrizable continuum is locally connected iff it has a convex generating metric.

Given such a metric continuum $\langle X, \varrho \rangle$ and $a, b \in X$ distinct, there is an isometry $f : [0, \varrho(a,b)] \to X$ with $f(0) = a$ and $f(\varrho(a,b)) = b$. If $c \in [a,b]$, then $c = f(t)$ for some $0 \leq t \leq \varrho(a,b)$; hence $\varrho(a,b) = t + (\varrho(a,b) - t) = \varrho(a,c) + \varrho(c,b)$. (So continuum betweenness implies metric betweenness.)

If A is the arc that is the image of the isometry f, then $\varrho^*(a, b) \leq \mathrm{diam}_\varrho(A) = \varrho(a, b)$; so $\varrho^* = \varrho$. This gives us

Proposition 9. Every locally connected metrizable continuum is straightenable.

Local connectedness is a very strong hypothesis, and we can improve on Proposition 9.

One important weakening of local connectedness is *aposyndesis*, due to F.B. Jones: given two points of the space, each is contained in the interior of a subcontinuum that excludes the other. Aposyndesis properly interpolates between local connectedness and Wilder-ness.

Proposition 10. Every aposyndetic metrizable continuum is straightenable.

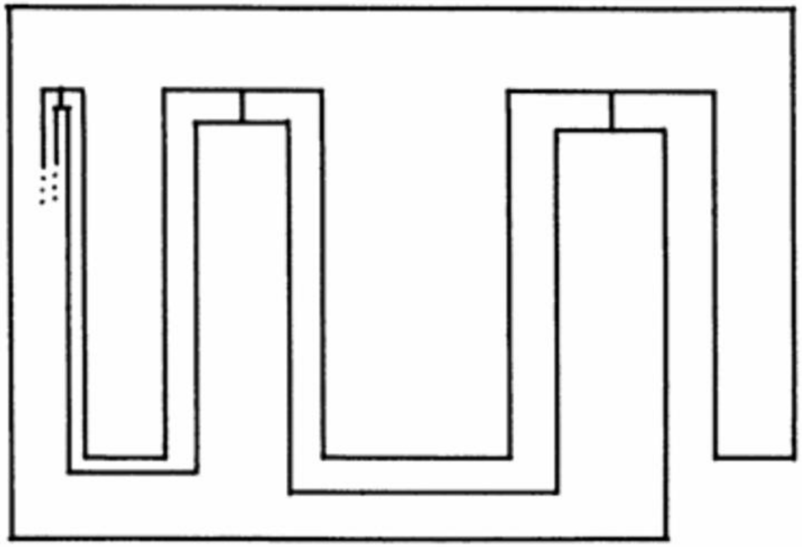
Proof. Assume X is not straightenable; we show it is not aposyndetic. For ϱ a generating metric, we have a sequence $\langle a_n \rangle$ that ϱ-converges to $a \in X$, but which does not ϱ^*-converge to a. In particular, for a tail of $\langle a_n \rangle$, we have $\text{diam}_\varrho([a_n, a]) \geq r$ for some fixed $r > 0$.

In each of these (compact) intervals, pick $b_n, c_n \in [a_n, a]$ such that $\mathrm{diam}_\varrho([a_n, a]) = \varrho(b_n, c_n)$. Then, using compactness and taking subsequences if necessary, we have $b, c \in X$ such that, relative to ϱ, $\langle b_n \rangle \to b$ and $\langle c_n \rangle \to c$. Since the distances $\varrho(b_n, c_n)$ are $\geq r$, so too is $\varrho(b, c)$.

Hence we have either $\varrho(a, b) \geq r/2$ or $\varrho(a, c) \geq r/2$; say it's the first case. Let $K \in \mathcal{K}(X)$ contain a in its interior K°. Then, for a tail of our original sequence, we have $a_n \in K^\circ$; hence $[a_n, a] \subseteq K$. In particular, $b_n \in K$; and this forces $b \in K$. Since $b \neq a$, we violate aposyndesis. $\square$

Example 7. The fans of Example 5 are Wilder continua that are non-straightenable.

Example 8. Thanks to a 1981 paper of D. Bellamy and L. Lum, we have a plane continuum that is non-aposyndetic, yet straightenable. The straightenability is due to the fact that the continuum is *cyclicly connected*; i.e., each two of its points ere contained in a simple closed curve. This makes the subcontinuum betweenness relation trivial (and hence the continuum is Wilder). Here's a picture.



The following diagram summarizes the placement of straightenability within the *locally connected–decomposable* spectrum.

LOCALLY CONNECTED

$\Downarrow \quad \not\Uparrow$

APOSYNDETIC

$\Downarrow \quad \not\Uparrow$

STRAIGHTENABLE

$\Downarrow \quad \not\Uparrow$

WILDER

$\Downarrow \quad \not\Uparrow$

DECOMPOSABLE

6. Ongoing work

- Characterize straightenability in purely topological language, along the lines of the Menger-Biing-Moise characterization of a continuum's admitting a convex generating metric iff the continuum is locally connected.

- The stretch of each generating metric for a hereditarily indecomposable continuum is an ultrametric (i.e., it satisfies the *strong triangle inequality* in which $+$ is replaced with max.) Is the converse true?

- A decomposable continuum can have at least one generating metric that stretches to an ultrametric; hence its stretch topology is zero-dimensional. If $\mathcal{P}$ is a topological property, call a continuum *stretch-*$\mathcal{P}$ if its stretch topology has property $\mathcal{P}$. Investigate this phenomonon for various properties (e.g., connectedness, local compactness, completeness, second countability, zero-dimensionality).

- Stretching metrics can be generalized to stretching uniformity bases. This opens up the study of straightenability for all continua. We beleve that aposyndetic continua are still straightenable in this setting, and straightenable continua are still Wilder. Are straightenable continua always decomposable? (*Wilder* $\Longrightarrow$ *decomposable* may not be true without *metrizable*.)

- We know that straightenable hereditarily unicoherent continua are decomposable; hence for example, the Stone-Čech remainder of the ray–being both indecomposable and hereditarily unicoherent–is non-straightenable. What about related continua, such as ultracoproducts?

THANK YOU!

Slides available at `https://www.mscsnet.mu.edu/∼paul/talks.html`.