

Stretching a Continuum Metric

(Joint with Daron Anderson and Aisling McCluskey)

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0. Quick Summary

This talk is about “stretching a metric to make intervals look straight.”

We look at continua through the lens of betweenness structures endowed with a metric (that may or may not generate the continuum topology).

We examine what it means for a subcontinuum betweenness interval to be “straight,” relative to a metric, as well as being “induced” by a metric in various ways.

We introduce the class of *straightenable continua* and examine its positioning relative to the classes of locally connected and of decomposable continua.

1. Metric Betweenness Structures

These are triples $\langle X, I, \rho \rangle$, where $\langle X, \rho \rangle$ is a metric space and I is a ternary relation on X , called a *betweenness relation*.

$I(a, c, b)$ is read, “point c lies between points a and b .”

Betweenness relations satisfy the following basic axioms.

- (*Inclusivity*) $I(a, a, b) \wedge I(a, b, b)$;
- (*Symmetry*) $I(a, c, b) \rightarrow I(b, c, a)$; and
- (*Uniqueness*) $I(a, c, a) \rightarrow a = c$.

For any betweenness relation I and points $a, b \in X$, the *betweenness interval bracketed by $\{a, b\}$* is the set

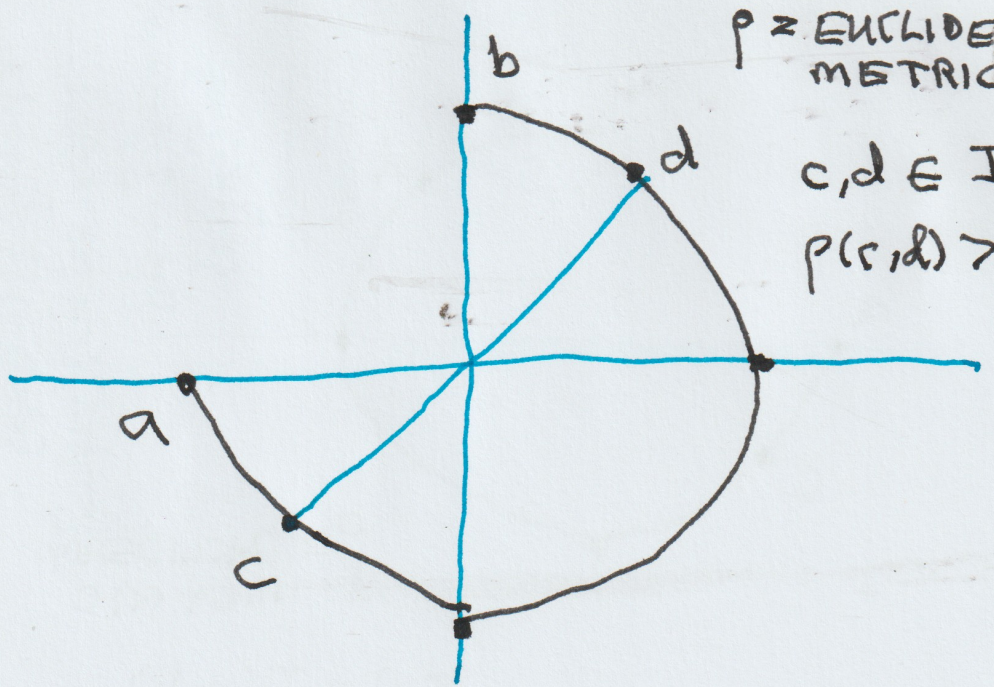
$$I(a, b) := \{x \in X : I(a, x, b) \text{ holds}\}.$$

So Inclusivity says $I(a, b) \supseteq \{a, b\}$, Symmetry says $I(a, b) = I(b, a)$, and Uniqueness says $I(a, a) = \{a\}$.

If $\langle X, I, \rho \rangle$ is a betweenness structure, intervals may look quite bendy, relative to the metric.

In the next figure, X is an arc that is wrapped about the unit circle in the Euclidean plane. $I(x, y)$ is the subarc with end points x, y .

The idea is that you can easily have points between a and b that are farther apart from each other than a is from b .



$\rho = \text{EUCLIDEAN METRIC}$

$c, d \in I(a, b)$

$\rho(c, d) > \rho(a, b)$

In this example, $X = \langle X, \mathcal{T} \rangle$ is a *continuum*; i.e., a connected compact Hausdorff space, and the metric ρ generates \mathcal{T} . (In general, it need not generate the topology.)

I is *subcontinuum betweenness*; i.e., $I(a, c, b)$ holds iff c lies in every subcontinuum that contains $\{a, b\}$.

With $\mathcal{K}(X)$ denoting the family of all subcontinua of X , we see that—in this case— $\mathcal{K}(X)$ comprises the subarcs of X ; in general we have

$I(a, b) = \bigcap \{K \in \mathcal{K}(X) : \{a, b\} \subseteq K\}$ (generally a compact set; connected precisely when the continuum is *hereditarily unicoherent*; i.e., two overlapping subcontinua have connected intersection).

Two additional conditions that hold for subcontinuum betweenness are:

- (*Transitivity*) $(I(a, c, b) \wedge I(a, d, c)) \rightarrow I(a, d, b)$; and
- (*Disjunctivity*) $I(a, d, b) \rightarrow (I(a, d, c) \vee I(b, d, c))$.

Both these conditions are easy to see; for example note that Disjunctivity in interval terms says that for any points a, b, c , $I(a, b) \subseteq I(a, c) \cup I(b, c)$. Now use the fact that the union of two overlapping subcontinua is a subcontinuum.

We call a betweenness relation *triodic* if it satisfies both Transitivity and Disjunctivity. (“In continua, especially the hereditarily unicoherent ones, triangles look like triods.”)

A consequence of triodicity in betweenness structures is a stronger form of Transitivity.

- (*Convexity*) $(I(a, c, b) \wedge I(a, d, b) \wedge I(c, e, d)) \rightarrow I(a, e, b)$.

When we think of *convexity* as modeled by betweenness intervals, this condition says that betweenness intervals themselves are *convex*.

If $\langle X, I, \varrho \rangle$ is a metric betweenness structure, we say that

- ϱ is *I-straightening* if $\varrho(c, d) \leq \varrho(a, b)$ whenever $c, d \in I(a, b)$.
- ϱ is *I-uniform* if for all $\varepsilon > 0$ there is $\delta > 0$ such that whenever $\varrho(a, b) \leq \delta$ we have $\varrho(c, d) \leq \varepsilon$ for all $c, d \in I(a, b)$.

Clearly I -straightening implies I -uniform; just let δ equal ε .

In the wrap-around arc example above, ρ is I -uniform without being I -straightening. (Indeed, $\rho(x, y) \leq \sqrt{2}\rho(u, v)$ whenever $x, y \in I(u, v)$.)

When I -uniformity fails for a metric, we can have a, b arbitrarily close, with the ratio $\rho(a, x)/\rho(a, b)$ being arbitrarily large as x ranges over the points of $I(a, b)$. (This is particularly true when I is subcontinuum betweenness in an indecomposable continuum.)

In any metric space $\langle X, \varrho \rangle$, we have the *metric betweenness* relation I_ϱ , given by

$$I_\varrho(a, c, b) \text{ iff } \varrho(a, b) = \varrho(a, c) + \varrho(c, b).$$

ϱ is I_ϱ -straightening; this is a consequence of Proposition 1.3 below.

This relation was first investigated by K. Menger in 1929. It clearly satisfies our three basic axioms, and Menger showed that it also satisfies Transitivity (but not necessarily Disjunctivity or Convexity).

It also clearly satisfies the following important condition:

- (*Antisymmetry*) $(I(a, c, b) \wedge I(a, b, c)) \rightarrow b = c.$

1.1 Remark. If we view betweenness as an indexed family of binary relations by rewriting $I(a, c, b)$ as $c \leq_a b$, then Transitivity says that each \leq_a is a transitive relation and Antisymmetry is the usual antisymmetry of binary relations.

(remark cont.) Antisymmetry in the context of subcontinuum betweenness may be rephrased as:

For any three distinct points $\{a, b, c\}$, there is a subcontinuum that contains a and exactly one of b, c .

This was first introduced independently by B. E. Wilder, who called it *Property C*. Wilder showed that Property C, for metrizable continua, is weaker than aposyndesis and stronger than decomposability.

We will refer to a continuum whose subcontinuum betweenness satisfies Antisymmetry as a *Wilder continuum*.

If $\langle X, I, \varrho \rangle$ is a metric betweenness structure, we say I is *induced* (resp., *subinduced*) by ϱ if $I = I_\varrho$ (resp., $I \subseteq I_\varrho$).

The following is immediate.

1.2 Proposition. Any betweenness relation subinduced by a metric satisfies Antisymmetry.

The next result relates subinducement and straightness.

1.3 Proposition. Let $\langle X, I, \rho \rangle$ be a metric betweenness structure where I is subinduced by ρ . Then ρ is I -straightening.

Proof. Fix $a, b \in X$. Then for any $x, y \in X$ we have both

$$(1) \quad \rho(x, y) \leq \rho(a, x) + \rho(a, y) \text{ and}$$

$$(2) \quad \rho(x, y) \leq \rho(x, b) + \rho(y, b);$$

and when we add these two inequalities we get

$$(3) \quad 2\rho(x, y) \leq (\rho(a, x) + \rho(x, b)) + (\rho(a, y) + \rho(y, b)).$$

$$(3) \quad 2\rho(x, y) \leq (\rho(a, x) + \rho(x, b)) + (\rho(a, y) + \rho(y, b)).$$

If I is subinduced by ρ and $x, y \in I(a, b)$, then the right-hand side of (3) equals $2\rho(a, b)$; hence ρ is I -straightening.

□

There is no converse to Proposition 1.3.

1.4 Example. A metric betweenness structure $\langle X, I, \rho \rangle$ where ρ is I -straightening but it is not the case that $I \subseteq I_\rho$:

Set $X = [0, 1] \subseteq \mathbb{R}$, where I is subcontinuum betweenness (so the betweenness intervals are the usual closed intervals of $[0, 1]$) and $\rho(x, y) := \sqrt{|x - y|}$. Then I_ρ is *trivial*—i.e., no interval contains more than two points—while I is not; so $I \not\subseteq I_\rho$.

On the other hand, the monotonicity of the square root function shows that ρ is I -straightening.

2. Stretch Metrics

Let $\langle X, I, \varrho \rangle$ be a metric betweenness structure. If ϱ fails to be I -straightening, then the diameter of $I(a, b)$ exceeds $\varrho(a, b)$. Define ϱ to be I -bounded if each diameter $\Delta_{\varrho}(I(a, b))$ is finite. In this case we define the I -stretch of ϱ to be

$$\varrho^I(a, b) := \Delta_{\varrho}(I(a, b)).$$

2.1 Proposition. Assume $\langle X, I, \varrho \rangle$ is a triodic metric betweenness structure such that ϱ is I -bounded. Then ϱ^I is a metric, $\varrho^I \geq \varrho$, and ϱ^I is I -straightening.

2.1 Proposition. Assume $\langle X, I, \varrho \rangle$ is a triodic metric betweenness structure such that ϱ is I -bounded. Then ϱ^I is a metric, $\varrho^I \geq \varrho$, and ϱ^I is I -straightening.

Proof. $\varrho^I \geq \varrho$ because each I -interval contains its bracket set (Inclusivity). If $\varrho^I(a, b) = 0$, then $I(a, b)$ has zero ϱ -diameter; hence $a = b$. Conversely, if $a = b$, then $I(a, b)$ is a singleton, by the Uniqueness axiom. Hence $\varrho^I(a, b) = 0$. ϱ^I is symmetric because of the Symmetry axiom.

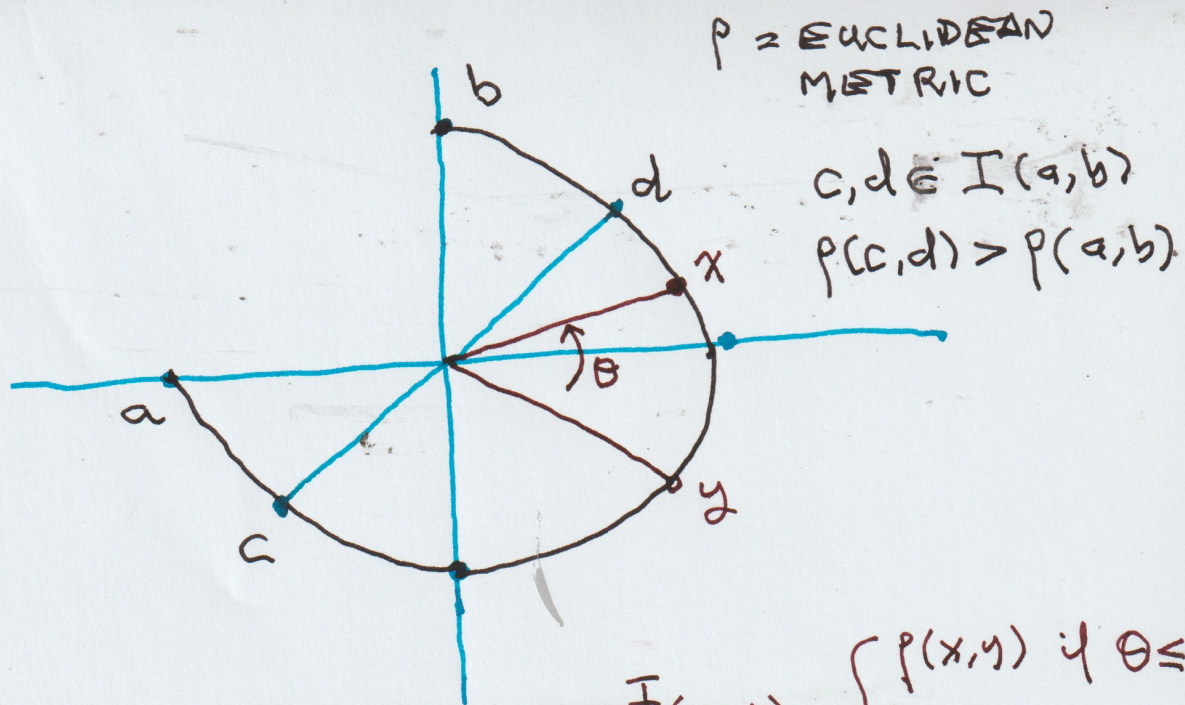
To see that the triangle inequality holds, let $a, b, c \in X$ be given. We have $\varrho^I(a, b) = \Delta_\varrho(I(a, b)) \leq \Delta_\varrho(I(a, c) \cup I(c, b))$, by Disjunctivity. It is well known that the diameter of the union of two intersecting sets is \leq the sum of the two diameters; hence the right-hand side is $\leq \Delta_\varrho(I(a, c)) + \Delta_\varrho(I(c, b)) = \varrho^I(a, c) + \varrho^I(c, b)$.

Finally, to see that ϱ^I is I -straightening, suppose $c, d \in I(a, b)$ are given. Then $\varrho^I(c, d) = \Delta_\varrho(I(c, d))$. But as noted above, Transitivity and Disjunctivity together imply Convexity, so we know $I(c, d) \subseteq I(a, b)$. Thus we immediately have $\varrho^I(c, d) \leq \varrho^I(a, b)$. \square

In the case I is subcontinuum betweenness in a metric continuum, triodicity holds and Proposition 2.1 immediately gives us the following.

2.2 Corollary. Let $\langle X, \varrho \rangle$ be a metric continuum, where $I = I_{\mathcal{K}}$ is subcontinuum betweenness. Then $\varrho^I \geq \varrho$ is a metric that is I -straightening.

Let's calculate ϱ^I in the wrap-around arc example above.



$$p^I(x, y) = \begin{cases} p(x, y) & \text{if } \theta \leq \pi \\ p(c, d) & \text{if } \theta \geq \pi \end{cases}$$

Some terminology: (1) If ρ and σ are two metrics on X , σ *topologically refines* ρ if the topology \mathcal{T}_σ generated by σ is finer than that generated by ρ . (I.e., $\mathcal{T}_\sigma \supseteq \mathcal{T}_\rho$.) If each metric refines the other, then they're termed *topologically equivalent*.

(2) σ *uniformly refines* ρ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\sigma(a, b) \leq \delta$ then $\rho(a, b) \leq \varepsilon$. If each metric uniformly refines the other, then they're termed *uniformly equivalent*.

Uniform refinement/equivalence implies topological refinement/equivalence; the following is well known.

2.3 Lemma. If two compact metrics are topologically equivalent then they're uniformly equivalent.

2.4 Proposition. Assume $\langle X, I, \varrho \rangle$ is a triodic metric betweenness structure such that ϱ is I -bounded. Then the stretch metric ϱ^I is uniformly equivalent to ϱ iff ϱ uniformly refines ϱ^I , iff ϱ is I -uniform.

Proof ϱ^I is a metric, thanks to Proposition 2.1. ϱ^I uniformly refines ϱ automatically, by virtue of the inequality $\varrho^I \geq \varrho$, so the first “iff” is immediate. The second “iff” is just a reworking of the relevant definitions. \square

In a metric betweenness structure $\langle X, I, \varrho \rangle$, the metric is *I -compact* if each I -interval is compact in the topology generated by ϱ .

2.5 Proposition. Assume $\langle X, I \rangle$ is a triodic betweenness structure, and let ϱ and σ be two metrics on X .

If σ is I -compact and topologically (resp., uniformly) refines ϱ , then ϱ is I -compact and σ^I topologically (resp., uniformly) refines ϱ^I .

Proof. We show the case where refinement is topological; the uniform case is handled similarly.

Assuming σ is I -compact and topologically refines ϱ , we know immediately that ϱ is I -compact. Hence both metrics are I -bounded, implying that both ϱ^I and σ^I are metrics (again by Proposition 2.1). It remains to show $\mathcal{T}_{\sigma^I} \supseteq \mathcal{T}_{\varrho^I}$.

For $a \in X$ and $r > 0$, $B_\varrho(a; r) := \{x \in X : \varrho(a, x) < r\}$ is the *open ϱ -ball of radius r , centered at a* . So to prove that σ^I topologically refines ϱ^I , it suffices to fix $a \in X$ and $r > 0$, and find $s > 0$ such that $B_{\sigma^I}(a; s) \subseteq B_{\varrho^I}(a; r)$.

Since σ topologically refines ϱ , we choose $s > 0$ such that $B_\sigma(a; s) \subseteq B_\varrho(a; \frac{r}{2})$.

Suppose $b \in X$ is such that $\sigma^I(a, b) < s$. We need to show that $\varrho^I(a, b) < r$.

For any $x, y \in I(a, b)$, we have $\sigma(a, x) \leq \sigma^I(a, b) < s$ and $\sigma(a, y) \leq \sigma^I(a, b) < s$; hence both $\varrho(a, x)$ and $\varrho(a, y)$ are $< \frac{r}{2}$. By the fact that ϱ is I -compact, we may choose $x, y \in I(a, b)$ such that $\varrho^I(a, b) = \varrho(x, y)$.

Hence $\varrho^I(a, b) = \varrho(x, y) \leq \varrho(a, x) + \varrho(a, y) < \frac{r}{2} + \frac{r}{2} = r$. \square

The following is an immediate consequence of Lemma 2.3, Proposition 2.5, and the fact that a sequence is Cauchy relative to one metric if it's Cauchy relative to a uniformly equivalent metric.

2.6 Corollary. Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum, with ϱ and σ two generating metrics for \mathcal{T} . Then, relative to subcontinuum betweenness, their respective stretch metrics are uniformly equivalent. In particular, their respective stretch metrics are either both complete or both incomplete.

This result (finally) allows us to make the following definition.

Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum. Then its *stretch topology*, denoted \mathcal{T}^I , is any one of the (equal) metric topologies \mathcal{T}_{ϱ^I} , where ϱ is a generating metric for \mathcal{T} .

The continuum $\langle X, \mathcal{T} \rangle$ is *straightenable* if $\mathcal{T}^I = \mathcal{T}$.

“Straightenable metric continua are the ones where you can stretch the metric in order to straighten the subcontinuum intervals and not break the topology.”

The stretch topology \mathcal{T}^I includes the continuum topology \mathcal{T} ; so $\langle X, \mathcal{T} \rangle$ is straightenable iff \mathcal{T}^I is compact. An obvious question is: *how close to being discrete can \mathcal{T}^I be?*

The answer is that while stretch topologies can be zero-dimensional; all of their nonempty open sets are equinumerous with the continuum itself.

2.7 Proposition. Let $\langle X, \mathcal{T} \rangle$ be a nondegenerate metrizable continuum, with $a \in X$ and $U \subseteq X$ a \mathcal{T}^I -open set containing a . Then there is a nondegenerate $K \in \mathcal{K}(X)$ with $a \in K \subseteq \underline{U}$.

Proof. Fix generating metric ϱ and $a \in U \subseteq X$, where U is ϱ^I -open. Then for some $r > 0$ we have the ϱ^I -open ball $B_{\varrho^I}(a; r)$ contained in U . Using ordinary boundary bumping for $\langle X, \varrho \rangle$, fix nondegenerate $K \in \mathcal{K}(X)$ such that

$$a \in K \subseteq B_{\varrho}(a; \frac{r}{3}).$$

Given any $b \in K$ we have $I(a, b) \subseteq K$. So pick $x, y \in I(a, b)$. Then

$$\varrho(x, y) \leq \varrho(a, x) + \varrho(a, y) < \frac{2r}{3};$$

$$\text{so } \varrho^I(a, b) = \Delta_{\varrho}(I(a, b)) \leq \frac{2r}{3} < r.$$

Hence $b \in B_{\varrho^I}(a; r)$. Since $b \in K$ is arbitrary, we have

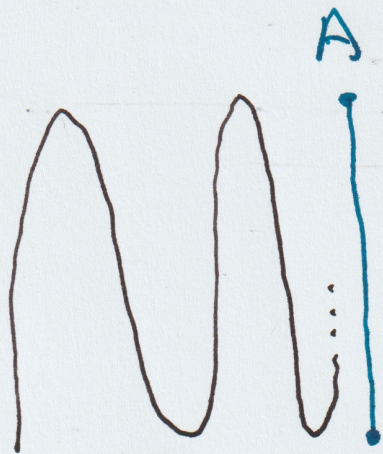
$$a \in K \subseteq B_{\varrho^I}(a; r) \subseteq U, \text{ as desired. } \square$$

3. Continua that are not Straightenable

We collect some facts about \mathcal{T}^I that distinguishes it from a continuum topology.

Let $\langle X, \mathcal{T} \rangle$ be any continuum, with K a proper nondegenerate subcontinuum of X . We call K a *spine* (a.k.a. *terminal subcontinuum*) of X if whenever $L \in \mathcal{K}$ intersects both K and $X \setminus K$, then $L \supseteq K$.

K is called a *semi-spine* of X if there is a point $v \in K$ (called a *vertex* of K) such that whenever $L \in \mathcal{K}(X)$ intersects both K and $X \setminus K$, then L contains v .



A is a spine



All maximal segments
are semi-spines; only
A has empty interior

3.1 Proposition. Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum.

(i) If K is a spine of X , then K is \mathcal{T}^I -open; hence K is a proper \mathcal{T}^I -clopen set and \mathcal{T}^I is a disconnected topology.

(ii) If K is a semi-spine of X , and $v \in K$ is a vertex of K , then every $a \in K \setminus \{v\}$ is in the \mathcal{T}^I -interior of K .

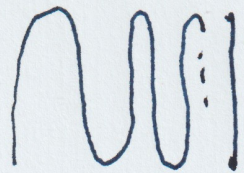
We'll prove (i); (ii) is similar.

Proof. Fix generating metric ϱ , spine K , and $a \in K$. Since K is nondegenerate, fix $p \in K \setminus \{a\}$ and $r = \varrho(a, p)$. Then $B_{\varrho^I}(a; r) \subseteq K$: otherwise we have some $b \in X \setminus K$, with $\Delta_{\varrho}(I(a, b)) = \varrho^I(a, b) < r$. But then every subcontinuum containing $\{a, b\}$ contains K ; hence $p \in I(a, b)$, and $\Delta_{\varrho}(I(a, b)) \geq r$. \square

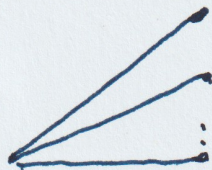
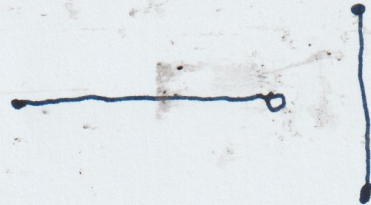
3.2 Corollary. Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum that contains either a spine or a nowhere dense semi-spine. Then X is unstraightenable. Moreover, if X has a spine, then \mathcal{T}^I fails to be connected.

3.3 Examples. (i) The usual $\sin \frac{1}{x}$ -continuum is unstraightenable because its limiting arc is a spine. Under the stretch topology, this continuum becomes the disjoint union of an arc and a ray.

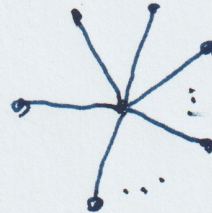
(ii) By a *fan* we mean the cone over an infinite Boolean space. If a is a limit point of that subspace (of end points) and v is the apex of the cone, then the arc with end points v and a is a semi-spine (v is a vertex) with empty interior. Hence no fan is straightenable. Under the stretch topology, this continuum becomes a hedgehog space. The hedgehog metric induces subcontinuum betweenness in the original fan, while not being a generating metric.



STRETCH



STRETCH



If $\langle X, \mathcal{T} \rangle$ is a continuum and $a \in X$, the *composant of X at a* is the union of all proper subcontinua of X that contain a . Each composant is a connected dense subset.

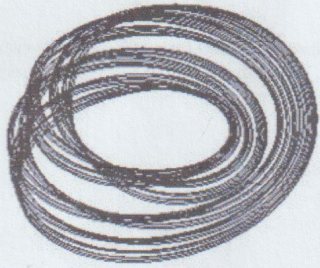
A continuum is *decomposable* if it is the union of two proper subcontinua; *indecomposable* otherwise. In indecomposable nondegenerate metrizable continua, the composants form a partition into $\mathfrak{c} := 2^{\aleph_0}$ dense subsets. In particular, the composants of such a continuum are not open in the continuum topology.

3.5 Proposition. Let $\langle X, \mathcal{T} \rangle$ be a metrizable continuum. Then every composant of X is \mathcal{T}^I -open. Consequently, every nondegenerate straightenable continuum is decomposable.

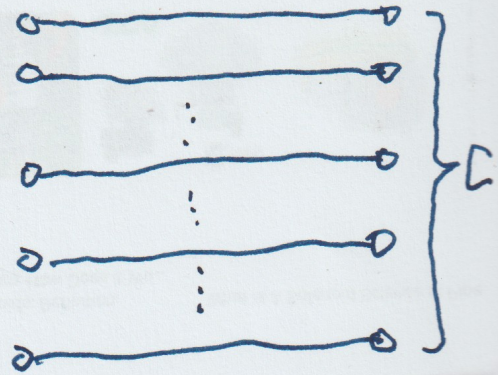
Proof. Given generating metric ϱ for \mathcal{T}^I , let $r = \Delta_\varrho(X)$. If $a, b \in X$ lie in separate composants of X then $\varrho(a, b) = r$. So given composant $C \subseteq X$ and $a \in C$, we have $B_{\varrho^I}(a; r) \subseteq C$. \square

From Proposition 3.5 we know that if $\langle X, \mathcal{T} \rangle$ is a non-degenerate indecomposable continuum, then its family of composants relative to \mathcal{T} becomes a decomposition of X into \mathfrak{c} -many \mathcal{T}^I -clopen sets.

3.6 Example. The *dyadic solenoid* is the limit of an inverse ω -indexed system of unit circles in the complex plane, with squaring as the bonding map. Because it is nondegenerate and indecomposable, it is nonstraightenable. Under the stretch topology, the composants become homeomorphic to the real line; hence the space is the free union (coproduct) of \mathfrak{c} -many copies of \mathbb{R} .



S-T-R-E-T-C-H



4. Continua that are Straightenable

By classical work of Menger, Bing and Moise, a metrizable continuum $\langle X, \mathcal{T} \rangle$ has a convex generating metric ϱ —i.e., where each nondegenerate I_ϱ -interval has more than two points—if and only if the continuum is locally connected.

Given such a metric continuum $\langle X, \varrho \rangle$ and $a, b \in X$ distinct, then, there is an isometry $f : [0, \varrho(a, b)] \rightarrow X$ with $f(0) = a$ and $f(\varrho(a, b)) = b$. If $I(a, c, b)$ holds, then $c = f(t)$ for some $0 \leq t \leq \varrho(a, b)$; hence $\varrho(a, b) = t + (\varrho(a, b) - t) = \varrho(a, c) + \varrho(c, b)$, and $I \subseteq I_\varrho$. (I is subinduced by ϱ .)

If A is the arc that is the image of the isometry f , then $\varrho^I(a, b) \leq \Delta_\varrho(A) = \varrho(a, b)$; so $\varrho^I = \varrho$.

In the event X is also hereditarily unicoherent, then $A = I(a, b)$; hence I is induced by ϱ .

These observations, together with Corollary 2.6, give us the following.

4.1 Corollary. Every locally connected metrizable continuum is straightenable.

Locally connected continua are straightenable in quite a dramatic way: their subcontinuum betweenness relations are even subinduced by their convex generating metrics—induced, in the hereditarily unicoherent case.

One important weakening of local connectedness is *aposyn-
desis*, due to F.B. Jones: given two points of the space, each is contained in the interior of a subcontinuum that excludes the other.

4.2 Proposition. Every aposyndetic metrizable continuum is straightenable.

Proof. Assume $\langle X, \mathcal{T} \rangle$ is not straightenable; we show it is not aposyndetic.

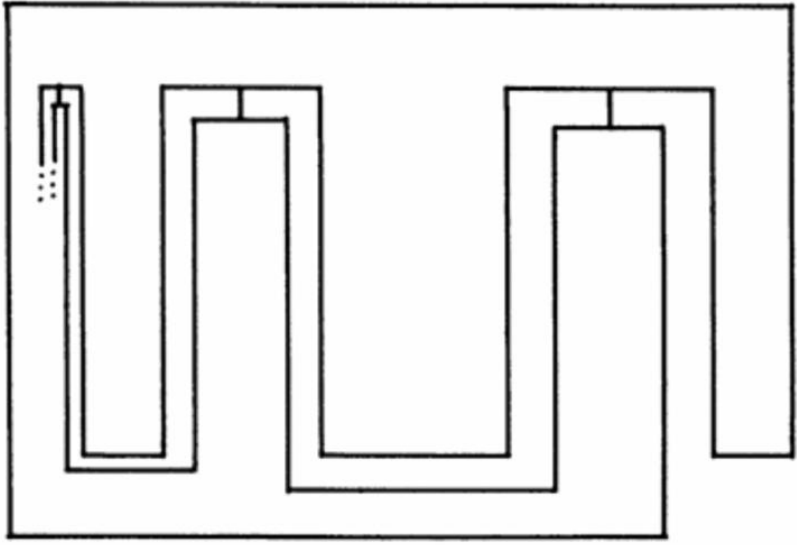
For ϱ a generating metric, we have a sequence $\langle a_n \rangle$ that ϱ -converges to $a \in X$, but which does not ϱ^I -converge to a . In particular, for a tail of $\langle a_n \rangle$, we have $\Delta_\varrho(I(a_n, a)) \geq r$ for some fixed $r > 0$.

In each of these (compact) intervals, pick $b_n, c_n \in I(a_n, a)$ such that $\Delta_\varrho(I(a_n, a)) = \varrho(b_n, c_n)$. Then, using compactness and taking subsequences if necessary, we have $b, c \in X$ such that, relative to ϱ , $\langle b_n \rangle \rightarrow b$ and $\langle c_n \rangle \rightarrow c$. Since the distances $\varrho(b_n, c_n)$ are $\geq r$, so too is $\varrho(b, c)$.

Hence we have either $\varrho(a, b) \geq r/2$ or $\varrho(a, c) \geq r/2$; say it's the first case. Let $K \in \mathcal{K}(X)$ contain a in its interior K° . Then, for a tail of our original sequence, we have $a_n \in K^\circ$; hence $I(a_n, a) \subseteq K$. In particular, $b_n \in K$; and this forces $b \in K$. Since $b \neq a$, we violate aposyndesis. \square

4.3 Remark. The fans of Example 3.3 (ii) are Wilder continua that are not straightenable.

4.4 Example. Thanks to a 1981 paper of D. Bellamy and L. Lum, we have a plane continuum that is non-aposyndetic, yet straightenable. The straightenability is due to the fact that the continuum is *cyclicly connected*; i.e., each two of its points are contained in a simple closed curve. This makes the subcontinuum betweenness relation trivial (and hence the continuum is Wilder).



4.5 Proposition. Every straightenable metrizable continuum is Wilder.

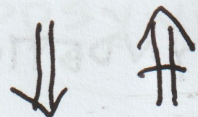
Proof. Assume $\langle X, \mathcal{T} \rangle$ is straightenable, but not Wilder. We derive a contradiction.

Being a non-Wilder continuum means that there are three distinct points $a, b, c \in X$ such that $I(a, b) = I(a, c)$. Let C be the union of all subcontinua of X that contain a and exclude b . Then, by assumption, we have $c \notin C$.

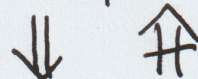
Let C' be the closure of C in X . If $b \notin C'$, then the subcontinuum C' is contained in the open set $X \setminus \{b\}$, and boundary bumping implies the existence of a subcontinuum $K \subseteq X \setminus \{b\}$, which properly contains C' . This contradicts the definition of C ; so we conclude $b \in C'$.

Let ρ be a generating metric for X , and fix a sequence $\langle b_n \rangle$ that consists of points of C and which ρ -converges to b . By straightenability, we know that $\langle b_n \rangle$ ρ^I -converges to b as well, implying that the diameters $\Delta_\rho(I(b_n, b))$ become arbitrarily small. Since $c \neq b$, there is an index m such that $c \notin I(b_m, b)$. Hence there is a subcontinuum K that contains $\{b_m, b\}$ but does not contain c . Since $b_m \in C$, there is a subcontinuum $L \subseteq C$ that contains $\{a, b_m\}$. In particular, we know $c \notin L$. Thus $K \cup L$ is a subcontinuum containing $\{a, b\}$ but not c , and we conclude $c \notin I(a, b)$, a contradiction. \square

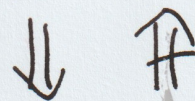
LOCALLY CONNECTED



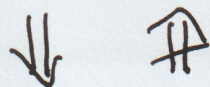
APOSYNDETIK



STRAIGHTENABLE



WILDER



DECOMPOSABLE

5. Ongoing Work

- Given a generating metric for a hereditarily indecomposable continuum, the stretch metric is an ultrametric. When is the subcontinuum betweenness relation induced by this metric (not just subinduced)?
- Stretching metrics can be generalized to stretching uniform bases. This opens up the study of straightenability for all continua. Aposyndetic continua are still straightenable in this setting, and straightenable continua are still Wilder. Are straightenable continua still decomposable? (We know they have decomposable subcontinua, but the number of composants is an issue.)

THANK YOU!

Slides available at
<https://www.mscsnet.mu.edu/~paul/talks.html>

(Celtic Tiger Meets Welsh Dragon)

All are invited to the 24th Galway Topology Colloquium,
09–10 July, 2026, University of Aberystwyth.

galwaytopology2026.org