Putnam 2020–21 A1

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How many positive integers N satisfy the three conditions

1. N is divisible by 2020.
2. N has at most 2020 decimal digits.
3. The decimal digits of N consist of a string of 1's followed by a string of 0's.

We have the prime number factorization $2020 = 20 * 101 = 2 * 2 * 5 * 101$. On Putnam exams one never knows what it may be good for.

After we did some work already, we decided to introduce abbreviation $1_m 0_n$ for a number which in decimal form consists of m ones followed by n zeros.

First we try to find *any* number satisfying the three conditions. With $5*2020 = 10100$ we see that $55 * 2020 = 111100 = 1_4 0_2$.

Addition and subtraction of multiples of 2020 are still multiples of 2020. Multiplication of a multiple of 2020 by any integer is still a multiple of 2020. These are all special cases of modular arithmetic. We apply these closure operations below.

Starting from $1_4 0_2$, we can add zeros to the right (multiplication by 10) to get distinct multiples of 2020, but don't cross the bound of at most 2020 digits. So we can add anywhere from 0 to $2020 - 6 = 2014$ zeros. This gives 2015 distinct solutions of the form $1_4 0_n$ with $2 \leq n \leq 2016$.

Addition $1_4 0_2 + 1_4 0_6 = 1_8 0_2$ gives another solution. Like above, we can add anywhere from 0 to $2020 - 10 = 2010$ zeros to the right and get 2011 distinct solutions of the form $1_8 0_n$ with $2 \leq n \leq 2012$.

We can repeat this process and get numbers of form $1_{4m} 0_n$ with $4m + n \leq 2020$ and $2 \leq n$, all satisfying the conditions. Let us count how many we have so far. For each $m \geq 1$ we have $2 \leq n \leq 2020 - 4m$, so $2020 - 4m - 1$ distinct numbers of form $1_{4m} 0_n$. Since $n \geq 2$, we have for m that $1 \leq 4m \leq 2020 - 2 = 2018$, or $\frac{1}{4} \leq m \leq 504 + \frac{1}{2}$, or easier still $1 \leq m \leq 504$. So our total is a sum of 504 numbers in arithmetic sequence

$$S = 2015 + 2011 + 2007 + \ldots + 11 + 7 + 3 \quad \text{(all 3 mod 4)}$$

We pretend that we don't recall the sum formula. So additionally write

$$S = 3 + 7 + 11 + \ldots + 2007 + 2011 + 2015$$

and add the two sequences term by term to get

$$2S = 2018 + 2018 + 2018 + \ldots + 2018 + 2018 + 2018 \quad \text{a total of 504 term}$$

So $2S = 2018 * 504$, thus $S = 1009 * 504 = 508536$. Below we show that there are no other solutions, which establishes that this is the final answer.

Let $1_m 0_n$ be a number satisfying the conditions. Since $1_m 0_n = 1_m * 10^n$ and 1_m is odd, number $1_m 0_n$ is exactly n times divisible by 2. Since 2020 is divisible by 2^2, we must have $n \geq 2$ for divisibility by 2020.

Let $N = 1_m 0_n$ be a number satisfying the conditions (so $n \geq 2$). There are q and r with $0 \leq r \leq 3$ such that $N = 1_{4q+r} 0_n$. It suffices to show that $r = 0$. Assume $r \geq 1$. Number $1_{4q} 0_n$ satisfies the conditions or equals 0, so $N - 1_{4q} 0_n = 1_r 0_{4q+n} = 1_r * 10^{4q+n}$ also satisfies the conditions. However, 101 divides 2020, but doesn't divide 10 or any of the numbers 1, 11, or 111. Contradiction. Thus $r = 0$.