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Wim Ruitenburg

Given integer $k \geq 0$, evaluate $\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}$.

A poorly formulated question. As some students pointed out, what does 'evaluate' mean? Anyway, as part of a discovery process, we compute the sum for a few small values of k with hopes of seeing a pattern. We use abbreviation s_k for the sum in the original question. Recall that $\binom{n}{m} = n!/(m!(n-m)!)$ whenever $0 \leq m \leq n$, and $\binom{n}{m} = 0$ whenever $0 \leq n < m$.

$$s_0 = 2^0\binom{0}{0} = 1$$

$$s_1 = 2^1\binom{1}{0} + 2^0\binom{2}{1} = 2 + 2 = 4 = 2^2$$

$$s_2 = 2^2\binom{2}{0} + 2^1\binom{3}{1} + 2^0\binom{4}{2} = 4 + 6 + 6 = 16 = 4^2 = 2^4$$

$$s_3 = 2^3\binom{3}{0} + 2^2\binom{4}{1} + 2^1\binom{5}{2} + 2^0\binom{6}{3} = 8 + 16 + 20 + 20 = 64 = 4^3 = 2^6$$

A pattern, suggesting that $s_k = 4^k = 2^{2k}$ for all $k \geq 0$. This 'conjecture' is likely a contribution towards the solution.

We suppose some familiarity with Pascal's Triangle and its correspondence with binomial coefficients. Motivated by illustrations farther below, we display the following top part of the Triangle:

$$\begin{array}{cccccccccccccc}
 & & & & & & 1 & & 0 & & 0 & & 0 & \\
 & & & & & 1 & & 1 & & 0 & & 0 & & 0 \\
 & & & & 1 & & 2 & & 1 & & 0 & & 0 & \\
 & & & 1 & & 3 & & 3 & & 1 & & 0 & & 0 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & 0 & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & 0 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 &
\end{array}$$

Each coefficient equals a binomial coefficient. For example, $5 = \binom{5}{1} = \binom{5}{4}$ and $20 = \binom{6}{3}$. We have the usual equations $\binom{n}{0} = 1$ and $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$, for all $n, k \geq 0$. The 0's to the right represent that $\binom{n}{m} = 0$ whenever $0 \leq n < m$.

Consider case $k = 3$. Let us rewrite $s_3 = 2^3 * 1 + 2^2 * 4 + 2^1 * 10 + 2^0 * 20 = 2^6$ where list $1, 4, 10, 20$ is a downward row in the Triangle picture. We know that the sum of each row of the Triangle is a power of 2, for example $1 + 6 + 15 + 20 + 15 + 6 + 1 = 2^6$. In this case we have $2^3 * 1 + 2^2 * 4 + 2^1 * 10 = 2 * (1 + 6 + 15)$. It would suffice to prove such an equation in general, where the left hand side is s_k, but with the last term removed. That is, it suffices prove (we divide both sides by 2)

$$\sum_{j=0}^{k-1} 2^{k-j-1}\binom{k+j}{j} = \sum_{j=0}^{k-1}\binom{2k}{j} \quad \text{if } k \geq 1$$

This equation itself is a further step towards a solution.

In a first attempt of a proof of the equation (by induction on k) we ran into trouble, which may have had something to do with $2k$ being even. So we decided to consider something more general below.

Define $f_m^n = \sum_{j=0}^{m}\binom{n}{j}$ for all $n, m \geq 0$. Recall that $\binom{n}{j}$ counts the number of subsets of size j of a set of size n. So f_m^n counts the number of subsets of size at most m of a set of size n. So $f_0^n = 1$ because of the empty subset, and $f_m^n = 2^n$ if $m \geq n$ because we count all subsets. Recall that in the end we are only interested in rewriting f_{k-1}^{2k} as

$$\sum_{j=0}^{k-1} 2^{k-j-1}\binom{k+j}{j} = f_{k-1}^{2k}$$

for $k \geq 4$ (we already checked the final equation for $k = 0, 1, 2, 3$)

Let us try to write f_m^{n+1} in terms of f_m^n, since such an equation may be helpful in proofs by induction on n. If $m = 0$, then $f_m^{n+1} = 1$. Suppose $m \geq 1$. Consider a subset $S \subseteq \{1, 2, 3, 4, \ldots, n\}$. If size $|S| \leq m - 1$, then we can extend S to a subset of $\{1, 2, 3, 4, \ldots, n, n+1\}$ of size at most m in exactly two ways, either add $n+1$ to S or don't. So $2f_{m-1}^n \leq f_m^{n+1}$. If $|S| = m$, then we count only S itself in f_m^{n+1}. Bigger sets than S don't count. So

$$f_m^{n+1} = 2f_{m-1}^n + \tbinom{n}{m} \quad \text{if } m \geq 1$$

If $m \geq 2$ and $n \geq 1$, then $f_{m-1}^n = 2f_{m-2}^{n-1} + \tbinom{n-1}{m-1}$. Substitute:

$$f_m^{n+1} = 2^2 f_{m-2}^{n-1} + 2^1 \tbinom{n-1}{m-1} + \tbinom{n}{m} \quad \text{if } m \geq 2 \text{ and } n \geq 1$$

Repeat.

$$f_m^{n+1} = 2^3 f_{m-3}^{n-2} + 2^2 \tbinom{n-2}{m-2} + 2^1 \tbinom{n-1}{m-1} + \tbinom{n}{m} \quad \text{if } m \geq 3 \text{ and } n \geq 2$$

We can repeat this p many times as long as $m \geq p$ and $n \geq p - 1$:

$$f_m^{n+1} = 2^p f_{m-p}^{n-p+1} + \textstyle\sum_{j=1}^{p} 2^{p-j} \tbinom{n-p+j}{m-p+j} \quad \text{if } m \geq p \text{ and } n \geq p - 1$$

We may suppose that $k \geq 1$. Set $n = 2k - 1$ and $m = p = k - 1$. So

$$f_{k-1}^{2k} = 2^{k-1} f_0^{k+1} + \textstyle\sum_{j=1}^{k-1} 2^{k-1-j} \tbinom{k+j}{j}$$

Since $f_0^{k+1} = 1 = \tbinom{k+0}{0}$, we thus have

$$f_{k-1}^{2k} = \textstyle\sum_{j=0}^{k-1} 2^{k-1-j} \tbinom{k+j}{j}$$