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Alice and Bob play a game of k pegs and n holes with $1 \leq k < n$. The holes are linearly ordered from 1 to n , and in the start situation the k pegs are in the left most holes 1 through k . With Alice beginning, the players take turns moving one peg to a position higher up. The game ends when there is no further such move possible. The one who cannot make a further move loses, the other is the winner. For which starting situations does Alice have a winning strategy?

Students were able to see a solution, but at times had difficulty writing one down. It turns out that it is easiest to first determine losing situations for Alice, and then show that all others are winning situations for Alice. When experimenting with small cases, we quickly see that Alice wins when $k = 1$ and $n > 1$ arbitrary, and there is a pattern for the case $k = 2$ and $n > 2$ (try $n = 3, 4, 5, 6$).

When both k and n are even, this is a losing situation for beginner Alice. To prove this, we show something stronger. We notice that during the game there is such a thing as the active zone which ranges from the left most hole with a peg to the right most empty hole. Empty holes farther left, and pegs at the end, are not part of any allowed move. These positions are essentially removed from the further progress of the game. We consider the case where the active zone has even length. Renumber its holes from 1 to $2m$ for some $m \geq 1$. Since $2m$ is even, we can partition the active zone into cells of form $\{2i - 1, 2i\}$ for $1 \leq i \leq m$. We add the condition that each cell either has no pegs or has two pegs. So there are no cells with a single peg. A situation where these two conditions hold we call bad. Beware that both conditions are satisfied in the start situation with k and n even. Suppose a player is given such a bad situation. At most what can be done is to move a peg from one of the filled cells to an empty cell farther to the right. So the player creates two cells each with one single peg. Now the opponent has a fatal response: Remove the remaining peg from the half filled cell on the left to the half filled cell farther to the right, thereby making the situation bad again. So no matter what legal move the player tries, the opponent has always such a response. Since in this way the opponent cannot lose, the player loses. For trivial reasons the strategies above remain valid when $k = 2m$ or $2m = 0$. Bad situations are losing situations.

It remains to show that all other starting situations, where k or n is odd, are winning situations.

Suppose we have a starting situation with k even and n odd. When Alice moves the peg from position 1 to position $k + 1$, she creates an active zone of even length $n - 1$, with an even number k of pegs on the far left in the active zone. A bad situation for Bob. Alice has a winning strategy.

Suppose we have a starting situation with k odd and n even. When Alice moves the peg from position 1 to position n , she creates an active zone of even length $n - 2$, with an even number $k - 1$ of pegs on the far left in the active zone. A bad situation for Bob. Alice has a winning strategy.

Suppose we have a starting situation with k odd and n odd. When Alice moves the peg from position k to position n , she creates an active zone of even length $n - 1$, with an even number $k - 1$ of pegs on the far left in the active zone. A bad situation for Bob. Alice has a winning strategy.