Putnam 2024 B1

Wim Ruitenburg

Given $k, n \in \mathbb{N}^+$, on an $n \times n$ grid we assign a function $f(i,j) = i + j - k$. Find all pairs (k,n) for which it is possible to have exactly one square in each row and column such that the values $f(i,j)$ in the selected n squares (i,j) are exactly $1, 2, \ldots, n$.

At least three students found solutions, sometimes multiple solutions, for all pairs of form $(k,n) = (k, 2k-1)$. No solutions were seen for other combinations (k,n). It turns out that the hard part is to show that no solutions exist for all other combinations of (k,n).

We can find a complete solution confirming the intuition of the students, by looking at the problem over a different angle. This is one of those AHA solutions, a flash of insight. We first illustrate the idea of the solution by looking at the example with $k = 3$ and $n = 5$. In the 5×5 picture below we have grid point $(1,1)$ in the bottom left corner, and grid point $(5,1)$ at the bottom right. Since $f(k,1) = k + 1 - k = 1$, in the picture $f(3,1) = 1$, that is where the letter A appears in the bottom row. All letters A appear exactly at the points (i,j) where $f(i,j) = 1$. Similarly, $f(i,j) = 2$ wherever there is a B, $f(i,j) = 3$ for each C, $f(i,j) = 4$ for D, and $f(i,j) = 5$ for E. All other fields are left blank, since we only pick one square from each of the anti-diagonals A through E. Additionally, the $n = 5$ chosen squares must be on different rows and columns. In that way the n chosen squares form the graph of a permutation σ in the usual way, a bijection from the set $\{1, 2, \ldots, n\}$ to itself where each chosen (i,j) means $\sigma(i) = j$.

C	D	E		
B	C	D	E	
A	B	C	D	E
	A	B	C	D
		A	B	C

Now the magic. We turn the picture counterclockwise one quarter of a circle:

		E	D	C
	E	D	C	B
E	D	C	B	A
D	C	B	A	
C	B	A		

For now we concentrate our descriptions on this re-oriented situation. We express matters in terms of n in general, so not only for $n = 5$. The chosen squares remain the graph of a permutation, say τ. The lettered diagonals turn into graphs of partial functions $f_c(i) = i + c$, where the constant c ranges over a list of n consecutive integers. There are integers $k_1, k_2, \ldots, k_n$ such that $\tau(i) = i + k_i$. Since the chosen squares are from distinct consecutive parallel diagonals, the set $\{k_1, k_2, \ldots, k_n\}$ equals a collection of n consecutive integers. The values $\tau(i)$ of permutation τ range over all integers of set $\{1, 2, \ldots, n\}$, so $\Sigma_{i=1}^n \tau(i) = \Sigma_{i=1}^n i$ (we don't need to know that this sum equals $n(n+1)/2$). Substitute: $\Sigma_{i=1}^n i + \Sigma_{i=1}^n k_i = \Sigma_{i=1}^n i$, so $\Sigma_{i=1}^n k_i = 0$. The only way by which the sum of consecutive integers sums to 0 is when, up to reordering, the list

$$k_1, k_2, \ldots, k_n \text{ equals } (-m), (-m+1), \ldots, (m-1), m \text{ for some } m \geqslant 0.$$

So $n = 2m + 1$. It is time to rotate the picture back one quarter of a circle clockwise. Before doing so, we prepare the conversion from m to k as follows. The bottom diagonal line of partial function $f_{-m}(x) = x - m$, illustrated by the letters A, reaches the right most column at position $(n, n-m) = (2m+1, m+1)$. When we rotate a quarter circle clockwise, position $(2m+1, m+1)$ becomes $(m+1, 1)$, exactly where $f(k, 1) = 1$. So $m + 1 = k$, and $n = 2k - 1$ for some $k > 0$. Now rotate back.

The grid must have size $(k, n) = (k, 2k - 1)$. In these situations, multiple students found multiple solutions. We give two, although one suffices.

On solution consists of choosing squares $(1, k), (2, k + 1), \ldots, (k, 2k - 1)$ plus $(k + 1, 1), (k + 2, 2), \ldots, (2k - 1, k - 1)$. The first list of k chosen squares covers all cases where $f(i, j) \equiv 1 \bmod 2$. The second list of $k - 1$ chosen squares covers all cases where $f(i, j) \equiv 0 \bmod 2$.

Another solution consists of choosing squares $(1, 2k - 1), (2, 2k - 3), \ldots, (k, 1)$ plus $(k + 1, 2k - 2), (k + 2, 2k - 4), \ldots (2k - 1, 2)$. The first list of k chosen squares covers all cases where $f(i, 2k - 2i + 1) = k - i + 1$ ranges over values k down to 1. The second list of $k - 1$ chosen squares covers all cases where $f(i, 4k - 2i) = 3k - i$ ranges over values $2k - 1$ down to $k + 1$.

We illustrate the two solutions for $(k, 2k - 1) = (4, 7)$:

D	E	F	●			
C	D	●	F	G		
B	●	D	E	F	G	
●	B	C	D	E	F	G
	A	B	C	D	E	●
		A	B	C	●	E
			A	●	C	D

●	E	F	G			
C	D	E	F	●		
B	●	D	E	F	G	
A	B	C	D	E	●	G
	A	●	C	D	E	F
		A	B	C	D	●
			●	B	C	D