

Putnam 2023 B1

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Consider an m -by- n grid of unit squares, indexed by (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i, j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i, j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1, j)$, $(i, j+1)$, and $(i+1, j+1)$ are unoccupied, then a legal move is to slide the coin from (i, j) to $(i+1, j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

Before we get to the answer with proof farther below, we insert the following introduction. Multiple students quickly mentioned binomial coefficients, and sometimes had an idea about the final answer. But no proof. Before sketching a proof, we insert a brief reminder about the binomial theorem and Pascal's Triangle and binomial coefficients. This reminder part has as main purpose clarity rather than precise correct proofs. The reminders are about results presumed known.

Consider the basic idea behind expanding a power like $(a+b)^3 = (a+b)(a+b)(a+b)$. Each term of the expansion is a product obtained by picking one of a or b from each of the components $(a+b)$ and multiply them. So there are 2 times 2 times 2 equals $2^3 = 8$ terms, and sum (think binary counting with 0s and 1s)

$$(a+b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb.$$

With $ab = ba$, we group products with equal numbers of a and b together and get the more well-known

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Compare this with Pascal's Triangle next, where each number is the sum of the two ones diagonally above it.

				1				
				1		1		
			1	2		1		
		1	3	3		1		
	1	4	6	4		1		
	1	5	10	10		5		1
1	6	15	20	15		6		1

and so on. When we associate going down left with the letter a , and going down right with the letter b , then each number in the Triangle counts the number of paths of a s and b s from the top to get to its location. For example, consider the left-most of the two numbers 10 in the Pascal Triangle. It takes three letters a and two letters b from the top to get there. There are $4 + 6 = 10$ such paths. Not surprisingly, we have

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

There is a different way to count this number 10. We count all lists of 5 letters consisting of a s and b s, of which exactly 3 are a s. This is like picking a sublist of 3 elements from a list of 5 elements, which can be done in 5 times 4 times 3 equals 60 ways because there are 5 ways to pick the first, then 4 ways to pick the second, and 3 ways to pick the third. However, each such sublist of 3 elements is picked in an ordered fashion. There are $3! = 1 * 2 * 3 = 6$ ways by which each sublist is chosen. We must divide the count 60 by $3! = 6$ because of this multiplicative over-count of each sublist of size 3. So the proper count is the binomial term

$$\binom{5}{3} = \frac{5 * 4 * 3}{1 * 2 * 3} = \frac{5!}{3!2!} = 10$$

The same applies to all numbers in the Pascal Triangle. Each can be replaced by its corresponding binomial coefficient

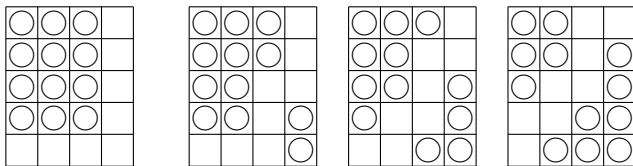
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

So the top part of the Triangle can be written as (recall that $0! = 1$)

$$\begin{array}{cccccccccccccccc} & & & & & & \binom{0}{0} & & & & & & & & & & \\ & & & & & \binom{1}{0} & \binom{1}{1} & & & & & & & & & & \\ & & & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & & & & & & & & & & \\ & & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & & & & & & & & & & & \\ & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & & & & & & & & & & & \\ \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} & & & & & & & & & & & \\ \binom{6}{0} & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & \binom{6}{6} & & & & & & & & & & \end{array}$$

Note that $\binom{5}{2} = \binom{5}{3}$. The Pascal Triangle is symmetric.

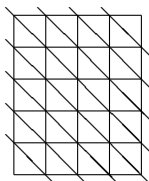
We return to answering the original question. The pictures below with $m = 5$ and $n = 4$ are for demonstration purposes only. The first illustrates the begin situation, the others are possible stages in the ‘game’.



We observe that the blank squares appear to represent all paths from the top right corner to the bottom left, as the Pascal Triangle does when rotated clockwise over angle $\pi/4$:

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 10 & 6 & 3 & 1 \\ 20 & 10 & 4 & 1 \\ 35 & 15 & 5 & 1 \end{array}$$

This indicates that we should look at the anti-diagonal lines as drawn over the board:



In the beginning each anti-line contains exactly one empty square, and the legal moves preserve this ‘exactly one’ property. So there is always one empty square on each anti-line. Additionally, empty squares on neighborly anti-lines touch at an edge, a property which holds in the beginning configuration and is preserved by the legal moves. So the empty squares always form a path from the top right square to the bottom left square, a path like in the Pascal Triangle. In the Pascal Triangle there are $\binom{m+n-2}{m-1}$ such paths. We claim that this binomial coefficient is the final answer.

It suffices to show that all paths occur in the game. We complete the proof by induction on $m + n$. The case for $m = n = 1$ is trivial. Suppose we know that it holds

for all boards of size (m, n) with $m + n = k$. Consider a new board of size (m, n) with $m + n = k + 1$. All paths that can be obtained in the game, start in the top right corner and end in the bottom left corner square $(m, 1)$. Each such path either passes through $(m - 1, 1)$ or passes through $(m, 2)$ before ending at $(m, 1)$. By supposition, in the first case each path from the top right to $(m - 1, 1)$ lies on a board of size $(m - 1, n)$ with $(m - 1) + n = k$, where all paths are obtainable. Its right-most path follows from the successive legal moves with coins $(m - 1, n - k)$ with $1 \leq k \leq n - 1$. By supposition, in the second case each path from the top right to $(m, 2)$ lies on a board of size $(m, n - 1)$ with $m + (n - 1) = k$, where all paths are obtainable. These include all possible paths on the board of size (m, n) with $m + n = k + 1$. By induction we get all paths for all boards.

The final answer is binomial coefficient $\binom{m+n-2}{m-1}$.