Putnam 2020–21 A1

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How many positive integers N satisfy the three conditions

1. N is divisible by 2020.
2. N has at most 2020 decimal digits.
3. The decimal digits of N consist of a string of 1's followed by a string of 0's.

We have the prime number factorization $2020 = 20 * 101 = 2 * 2 * 5 * 101$. On Putnam exams one never knows what it may be good for.

After we did some work already, we decided to introduce abbreviation 1_m0_n for a number which in decimal form consists of m ones followed by n zeros.

First we try to find *any* number satisfying the three conditions. With $5*2020 = 10100$ we see that $55 * 2020 = 111100 = 1_40_2$.

Addition and subtraction of multiples of 2020 are still multiples of 2020. Multiplication of a multiple of 2020 by any integer is still a multiple of 2020. These are all special cases of modular arithmetic. We apply these closure operations below.

Starting from 1_40_2, we can add zeros to the right (multiplication by 10) to get distinct multiples of 2020, but don't cross the bound of at most 2020 digits. So we can add anywhere from 0 to $2020 - 6 = 2014$ zeros. This gives 2015 distinct solutions of the form 1_40_n with $2 \leq n \leq 2016$.

Addition $1_40_2 + 1_40_6 = 1_80_2$ gives another solution. Like above, we can add anywhere from 0 to $2020 - 10 = 2010$ zeros to the right and get 2011 distinct solutions of the form 1_80_n with $2 \leq n \leq 2012$.

We can repeat this process and get numbers of form $1_{4m}0_n$ with $4m + n \leq 2020$ and $2 \leq n$, all satisfying the conditions. Let us count how many we have so far. For each $m \geq 1$ we have $2 \leq n \leq 2020 - 4m$, so $2020 - 4m - 1$ distinct numbers of form $1_{4m}0_n$. Since $n \geq 2$, we have for m that $1 \leq 4m \leq 2020 - 2 = 2018$, or $\frac{1}{4} \leq m \leq 504 + \frac{1}{2}$, or easier still $1 \leq m \leq 504$. So our total is a sum of 504 numbers in arithmetic sequence

$$S = 2015 + 2011 + 2007 + \ldots + 11 + 7 + 3 \quad \text{(all 3 mod 4)}$$

We pretend that we don't recall the sum formula. So additionally write

$$S = 3 + 7 + 11 + \ldots + 2007 + 2011 + 2015$$

and add the two sequences term by term to get

$$2S = 2018 + 2018 + 2018 + \ldots + 2018 + 2018 + 2018 \quad \text{a total of 504 terms}$$

So $2S = 2018 * 504$, thus $S = 1009 * 504 = 508536$. Below we show that there are no other solutions, which establishes that this is the final answer.

Let 1_m0_n be a number satisfying the conditions. Since $1_m0_n = 1_m * 10^n$ and 1_m is odd, number 1_m0_n is exactly n times divisible by 2. Since 2020 is divisible by 2^2, we must have $n \geq 2$ for divisibility by 2020.

Let $N = 1_m0_n$ be a number satisfying the conditions (so $n \geq 2$). There are q and r with $0 \leq r \leq 3$ such that $N = 1_{4q+r}0_n$. It suffices to show that $r = 0$. Assume $r \geq 1$. Number $1_{4q}0_{n+r}$ satisfies the conditions or equals 0, so $N - 1_{4q}0_{n+r} = 1_r0_n = 1_r * 10^n$ also satisfies the conditions. However, 101 divides 2020, but doesn't divide 10 or any of the numbers 1, 11, or 111. Contradiction. Thus $r = 0$.