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In our own words, let $S \subseteq \mathrm{N}^+$ be the least set satisfying properties

S1. $2 \in S$.

S2. $n^2 \in S$ implies $n \in S$, for all n.

S3. $n \in S$ implies $(n+5)^2 \in S$, for all n.

Determine the complement set $\mathrm{N}^+ \setminus S$.

Combining properties S3 and S2, we have that

S4. $n \in S$ implies $n + 5 \in S$, for all n.

So, for example, $2, 7, 12, 17, 22, 27, \ldots \in S$. More precisely, we have

S5. $\{m \in \mathrm{N}^+ \mid m \equiv 2 \bmod 5\} \subseteq S$.

By S3 we have $(2+5)^2 = 49 \in S$. Since $49 \equiv 4 \bmod 5$ we have

S6. $\{m \in \mathrm{N}^+ \mid m \equiv 4 \bmod 5 \text{ and } m \geq 49\} \subseteq S$.

We have $(49+5)^2 \equiv 4^2 \equiv 1 \bmod 5$. So with S3 we have that there is some $A \in S$ with $A \equiv 1 \bmod 5$. So also

S7. $\{m \in \mathrm{N}^+ \mid m \equiv 1 \bmod 5 \text{ and } m \geq A\} \subseteq S$.

It is time for the 'harvest', We have $1 \equiv 6 \equiv 6^2 \equiv 6^4 \equiv 6^8 \equiv \ldots \bmod 5$. Some 2^n power of 6 will be beyond A. So by repeated application of S2 we get $6 \in S$. So also

S8. $\{m \in \mathrm{N}^+ \mid m \equiv 1 \bmod 5 \text{ and } m \geq 6\} \subseteq S$.

$3^4 \equiv 1 \bmod 5$ plus S2 implies

S9. $\{m \in \mathrm{N}^+ \mid m \equiv 3 \bmod 5\} \subseteq S$.

$4^2 \equiv 1 \bmod 5$ plus S2 implies

S10. $\{m \in \mathrm{N}^+ \mid m \equiv 4 \bmod 5\} \subseteq S$.

The elements left are

S11. $\{1\} \cup \{m \in \mathrm{N}^+ \mid m \equiv 0 \bmod 5\}$.

We claim that these form $\mathrm{N}^+ \setminus S$. Obviously $1 \notin S$ since even the only remotely viable S2 concludes $1 \in S$ from supposition $1^2 = 1 \in S$. Both rules S2 and S3 suppose an element 0 mod 5 to construct an element 0 mod 5.