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Alice and Bob play a game in which they take turns choosing integers in the range 1 through n. Bob starts by selecting goal 'odd' or 'even'. Then Alice and Bob alternate by choosing a new integer in the range 1 through n, beginning with Alice at turn 1 and necessarily ending after n turns. Bob wins if the size of the 'diagonal' set $\{k \mid k$ was chosen by Alice or Bob at turn $k\}$ agrees with Bob's goal. For which positive n does Bob have a winning strategy?

The ideas behind the proof below are mostly from Daniel Matz.

With each game Alice and Bob build a permutation π on $\{1, 2, 3, \ldots, n\}$, where Bob wins if the size of diagonal set $D = \{k \mid \pi(k) = k\}$ agrees with his goal. After some experimentation for $n = 1, 2, 3, 4$, we conclude that the answer is that Bob can always win. We first consider the winning strategy for when n is even, and then the winning strategy for when n is odd.

The case for n even, which happens to be the relatively easy case. Bob selects goal 'even'. Since n is even, we can partition the set $\{1, 2, 3, \ldots, n\}$ into a collection of cells $\{2i - 1, 2i\}$ for $1 \leq i \leq n/2$. We call an element of one of these cells the *sibling* of the other element of this same cell. Once Alice starts choosing elements from $\{1, 2, 3, \ldots, n\}$, Bob follows by choosing its sibling. So Alice must choose from an unused cell, while Bob makes that a cell use is exhausted. Alice only moves for odd turns k, so $k = 2i - 1$ for some i. Suppose Alice picks $\pi(k) = k$, which adds k to the diagonal set D. So Bob at turn $k + 1 = 2i$ picks sibling $\pi(k + 1) = 2i = k + 1$. So Bob adds $k + 1$ to the diagonal set D, whose size therefore increases by $2 \equiv 0 \bmod 2$. Otherwise, suppose Alice picks $\pi(k) \neq k$. If $\pi(k) = 2j$ even, then Bob picks sibling $\pi(k + 1) = \pi(2i) = 2j - 1$ off the diagonal. So D is left unchanged. If $\pi(k) = 2j - 1$ odd with $i \neq j$, then Bob picks sibling $\pi(k + 1) = \pi(2i) = 2j$ off the diagonal. So D is left unchanged. Thus the size of D remains at 0 mod 2 even.

The case for n odd. Bob selects goal 'odd'. In our attempt to simplify the description of a winning strategy for Bob, we introduce a subclass of situations we call BWA (Bob Wins All) for which there is a special winning strategy for Bob.

Situation BWA means that two conditions hold, namely

1. So far there are an odd number of elements on the diagonal under construction.
2. It is Bob's turn to make a move.

In case of BWA, Bob has the following strategy which prevents any additions to diagonal set D under construction, and so keeps its size odd. Bob at his turn k faces one of two possible scenarios. Suppose $k + 1$ is still available. Then Bob picks $\pi(k) = k + 1$ off the diagonal. Otherwise, suppose that $k + 1$ was chosen by Alice or Bob at an earlier turn. Since n is odd and k is even, Bob's turn isn't the last one. Bob has at least two choices, one of which must be some $m \neq k$. So Bob picks $\pi(k) = m$ off the diagonal. Both scenarios lead to no additions to D under construction, and when it is the turn of Alice, she must pick $\pi(k + 1) \neq k + 1$ which also leaves the size of D unchanged. Repeat. At the next turn $k + 2$, if there is one, we are back in situation BWA. The size of D under construction doesn't change, so remains odd.

Back to the general question with n odd. We complete the proof by induction on odd n, that is, we may suppose that we have an odd value $n \geq 5$ so that all cases less than n have been settled as winnable by Bob following the earlier claims (cases $n = 1$ and $n = 3$ are trivial). At each of her turns, Alice must avoid creating a situation BWA. For example, Alice shouldn't pick $\pi(1) = 1$. There are two scenarios. • First, suppose

Alice picks $\pi(1) = m$ with $m \geq 3$ odd. Then Bob picks $\pi(2) = 2$ making the size of diagonal D under construction odd. To avoid BWA, Alice must pick $\pi(3) = 3$ if possible (so if $m \neq 3$). Then Bob pick $\pi(4) = 4$ making the size of D under construction odd again. And so on for a while. At her turn m, Alice can only pick $\pi(m) \neq m$, and so leaves a situation BWA. Alice looses. • Second, suppose Alice picks $\pi(1) = m$ with $m \geq 2$ even. We choose to treat the case $\pi(1) = 2$ separately. In that case Bob moves $\pi(2) = 1$, which leaves the size of D under construction equal to the even number 0. The remainder of the game is completely done with respect to the linearly ordered subset $\{3, 4, 5, \ldots, n\}$. When we 'move' each pair (k_1, k_2) of $\{3, 4, 5, \ldots, n\}^2$ to $(k_1 - 2, k_2 - 2)$, we move the square to $\{1, 2, 3, \ldots, n-2\}^2$ and send diagonal entries to diagonal entries. Up to this renaming, this is the same game, but now for the smaller odd number $n - 2$. By hypothesis, Bob can win this game with goal 'odd', and therefore win the original game with odd number n with goal 'odd'. Next the more general $\pi(1) = m$ with $m \geq 4$ even. Bob picks $\pi(2) = 2$. To avoid BWA, Alice picks $\pi(3) = 3$. If $m \geq 6$, then Bob picks $\pi(4) = 4$ and makes the size of D under construction odd again. So Alice has to pick $\pi(5) = 5$ to avoid BWA. And so on until Alice picks $\pi(m - 1) = m - 1$. Now the magic: Bob picks $\pi(m) = 1$. So far Alice and Bob have constructed a permutation on subset $\{1, 2, 3, \ldots, m\}$ with an even number of elements on the diagonal D under construction (in fact, $\pi(1) = m$ and $\pi(m) = 1$ and $\pi(k) = k$ for $2 \leq k \leq m - 1$). The remainder of the game is completely done with respect to the linearly ordered subset $\{m + 1, m + 2, m + 3, \ldots, n\}$ which, up to renaming $(k_1, k_2) \mapsto (k_1 - m, k_2 - m)$, is the same game, but for the smaller odd number $n - m$. By hypothesis, Bob can win this game with goal 'odd', and therefore win the original game with odd number n with goal 'odd'. Thus, by induction, Bob wins for all odd n with goal 'odd'.