

Putnam 2021 A1

WIM RUITENBURG

In our own words, we start at the origin $(0, 0)$ of the integer grid $\mathbb{Z} \times \mathbb{Z}$. At all times we are allowed to hop a Euclidean distance of 5 in all directions, as long as we arrive at another point of the grid. What is the smallest number of hops to get to $(2021, 2021)$?

The only possible hops are of kinds $(5, 0)$, $(4, 3)$, $(3, 4)$, $(0, 5)$, $(-3, 4)$, $(-4, 3)$, $(-5, 0)$, $(-4, -3)$, $(-3, -4)$, $(0, -5)$, $(3, -4)$, and $(4, -3)$. We expect to only need $(5, 0)$, $(4, 3)$, $(3, 4)$, and $(0, 5)$, exactly the hops without a minus sign $-$.

One may read the quest as owing points ('money') to two parties, each to the amount of 2021, for a total of 4042. We wish to cancel the debt in as few installments as possible, where each hop represents an installment, some to one party, some as a combination to both. The largest possible installment is 7 points by hops $(4, 3)$ or $(3, 4)$.

Now $4042/7 = 577 + 3/7$, so the number of installments (hops) is at least 578. We are done when we give a solution in 578 hops, thereby proving that 578 is the smallest number. When we perform 288 times a hop $(4, 3)$ and 288 times a hop $(3, 4)$, we are after 576 hops in total at point $(2016, 2016)$. Perform hops $(5, 0)$ and $(0, 5)$ and arrive at $(2021, 2021)$ in the minimum of 578 hops.