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Determine all $n \in \mathbb{N}^+$ for which there exist $a, b, c \in \mathbb{N}^+$ satisfying $2a^n + 3b^n = 4c^n$.

First we try to find any solution. When we set $n = 1$ we get equation $2a + 3b = 4c$, for which we can find multiple solutions. One suffices, $(a, b, c) = (3, 2, 3)$. So for $n = 1$ there is a solution.

From now on we may suppose $n \geqslant 2$, because we settled case $n = 1$. There are no immediate clear solutions. The presence of coefficients divisible by 2 suggests arithmetic modulo 2. An integer solution (a, b, c) implies that $3b^n \equiv 0 \bmod 2$. So b is divisible by 2. There is $b_1 \in \mathbb{N}^+$ such that $b = 2b_1$. Back to the original equation, we have $2a^n + 3 * 2^n b_1^n = 2^2 c^n$, so also

$$a^n + 3 * 2^{n-1} b_1^n = 2c^n.$$

Since $n - 1 \geqslant 1$, this gives $a^n \equiv 0 \bmod 2$. So a is divisible by 2. There is $a_1 \in \mathbb{N}^+$ such that $a = 2a_1$. Back to the original equation, we have $2^{n+1} a_1^n + 3 * 2^n b_1^n = 2^2 c^n$, so also

$$2^{n-1} a_1^n + 3 * 2^{n-2} b_1^n = c^n.$$

To repeat the case above for a even, but now to have c even, we need that $n - 2 \geqslant 1$. So we temporarily abandon the case $n = 2$ and suppose the stronger $n \geqslant 3$. This gives $c^n \equiv 0 \bmod 2$. There is $c_1 \in \mathbb{N}^+$ such that $c = 2c_1$. Back to the original equation, we have $2^{n+1} a_1^n + 3 * 2^n b_1^n = 2^{n+2} c_1^n$, so also

$$2a_1^n + 3b_1^n = 4c_1^n.$$

Aha, a ‘smaller’ solution to the original equation! So suppose $n \geqslant 3$, and assume that there is a solution (a, b, c). We may suppose that value $a + b + c$ is minimal among all solutions for this n, since minimal ones must exist. The above computation shows that all of a, b and c are even, and $(a/2, b/2, c/2)$ is also a solution for this n, with $a/2 + b/2 + c/2 < a + b + c$. So there is no ‘minimal’ solution, contradiction. The assumption is false. Thus for all $n \geqslant 3$ there are no solutions.

We are left with solving case $n = 2$ and equation $2a^2 + 3b^2 = 4c^2$. We don’t try arithmetic modulo 2 again. Instead, the presence of a 3 suggests arithmetic modulo 3. Since $2 \equiv -1 \bmod 3$ and $4 \equiv 1 \bmod 3$ we have $-a^2 \equiv c^2 \bmod 3$, so also

$$a^2 + c^2 \equiv 0 \bmod 3.$$

Each $x \in \mathbb{N}^+$ equals 0, 1, or 2 modulo 3, so x^2 equals $0^2 \equiv 0 \bmod 3$ or $1^2 \equiv 1 \bmod 3$ or $2^2 \equiv 1 \bmod 3$. So both a^2 and c^2 can at most be 0 or 1 modulo 3. The only combination for which the equation modulo 3 can work is for $a^2 \equiv c^2 \equiv 0 \bmod 3$. So a and c are both divisible by 3. There are $a_1, c_1 \in \mathbb{N}^+$ such that $a = 3a_1$ and $c = 3c_1$. Back to the original equation, we have $2 * 3^2 a_1^2 + 3b^2 = 4 * 3^2 c_1^2$, so also

$$2 * 3a_1^2 + b^2 = 4 * 3c_1^2.$$

This gives $b^2 \equiv 0 \bmod 3$. So b is divisible by 3. There is $b_1 \in \mathbb{N}^+$ such that $b = 3b_1$. Back to the original equation, we have $2 * 3^2 a_1^2 + 3^3 b_1^2 = 4 * 3^2 c_1^2$, so also

$$2a_1^2 + 3b_1^2 = 4c_1^2.$$

Aha, a ‘smaller’ solution to the original equation! So suppose $n = 2$, and assume that there is a solution (a, b, c). We may suppose that value $a + b + c$ is minimal among all solutions since minimal ones must exist. The above computation shows that all of a, b and c are divisible by 3, and $(a/3, b/3, c/3)$ is also a solution, with $a/3 + b/3 + c/3 < a + b + c$. So there is no ‘minimal’ solution, contradiction. The assumption is false. Thus for $n = 2$ there are no solutions.