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Wim Ruitenburg

For all $x \in \mathbb{R}^+$ define $g(x) = \lim_{r\to 0}((x+1)^{r+1} - x^{r+1})^{1/r}$. Find $\lim_{x\to\infty} \frac{g(x)}{x}$.

We ignore professors who walk by and mention Euler's e. Instead, we adapt the equations step by step so that each time the problem looks easier to solve.

For all positive reals a, b and all c we have $(ab)^c = a^c b^c$. So $(x+1)^{r+1} - x^{r+1} = x^{r+1}(1+1/x)^{r+1} - x^{r+1} = x^r x((1+1/x)^{r+1} - 1)$. So

$((x+1)^{r+1} - x^{r+1})^{1/r} = x(x((1+1/x)^{r+1} - 1))^{1/r}$, and
$\frac{g(x)}{x} = \lim_{r\to 0}(x((1+1/x)^{r+1} - 1))^{1/r}$

When x goes to infinity ∞, quantity $1/x$ goes towards 0 from the positive side. Set $y = 1/x$. We get

$$\lim_{x\to\infty} \frac{g(x)}{x} = \lim_{y\downarrow 0} yg(1/y) = \lim_{y\downarrow 0} \lim_{r\to 0} (\frac{(1+y)^{r+1} - 1}{y})^{1/r}$$

We may suppose that y is small, say $0 < y < \frac{1}{10}$. Then $(1+y)^{r+1} = \sum_{i\geqslant 0} \binom{r+1}{i} y^i = 1 + (r+1)y + \frac{(r+1)r}{2}y^2 + \frac{(r+1)r(r-1)}{6}y^3 + \ldots$. So after canceling the first 1, we have

$yg(1/y) = \lim_{r\to 0}(\frac{\sum_{i\geqslant 1}\binom{r+1}{i}y^i}{y})^{1/r} = \lim_{r\to 0}(\sum_{i\geqslant 1}\binom{r+1}{i}y^{i-1})^{1/r} =$
$\lim_{r\to 0}(1 + r(1 + \frac{r+1}{2}y + \frac{(r+1)(r-1)}{6}y^2 + \ldots))^{1/r}$

We may suppose that r is also small, say $0 < |r| < \frac{1}{10}$. So

$$h_r(y) = \frac{r+1}{2}y + \frac{(r+1)(r-1)}{6}y^2 + \ldots = y(\frac{r+1}{2} + \frac{(r+1)(r-1)}{6}y + \ldots)$$

is such that $|h_r(y)| < 5y$ (the 5 is chosen generously), and

$$yg(1/y) = \lim_{r\to 0}(1 + r(1 + h_r(y)))^{1/r}$$

So (we get the same limit regardless of whether $r > 0$ or $r < 0$) we have

$$\lim_{r\to 0}(1 + r(1-5y))^{1/r} \leqslant yg(1/y) \leqslant \lim_{r\to 0}(1 + r(1+5y))^{1/r}$$

We recall from calculus (hopefully from analysis as well) that $\lim_{r\to 0}(1+rc)^{1/r} = e^c$. As a reminder, take the natural logarithm $\ln((1+rc)^{1/r}) = \frac{1}{r}\ln(1+rc)$ and notice that $\lim_{r\to 0}$ of the expression looks like a derivative, and so on. So

$$e^{1-5y} \leqslant yg(1/y) \leqslant e^{1+5y}$$

Thus $\lim_{x\to\infty} \frac{g(x)}{x} = \lim_{y\downarrow 0} yg(1/y) = e$.