Putnam 2022 B1

Wim Ruitenburg

Given integer coefficient polynomial $P(x) = a_1x + a_2x^2 + \ldots + a_nx^n$ with a_1 odd, and $e^{P(x)} = b_0 + b_1x + b_2x^2 + \ldots$, prove that all b_k are nonzero.

To get some hint about the coefficients b_k, we compute the first few terms. Recall that $e^y = \sum_{k\geqslant 0} \frac{y^k}{k!} = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \ldots$. So

$$e^{P(x)} = 1 + (a_1x + a_2x^2 + \ldots) + \frac{(a_1x+a_2x^2+\ldots)^2}{2} + \ldots =$$
$$1 + a_1x + (a_2 + \frac{a_1^2}{2})x^2 + \ldots$$

So $b_0 = 1$ and $b_1 = a_1$ and $b_2 = \frac{2a_2+a_1^2}{2}$. Based on this flimsy evidence we speculate that for all k we have $b_k = \frac{c_k}{k!}$ with c_k an odd integer, that is, $k!\,b_k$ is always an odd integer. Such expressions occur with repeated differentiation as follows. Write $f(x)$ as short for $e^{P(x)}$. We have

$$f(x) = f^{(0)}(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots$$
$$f'(x) = f^{(1)}(x) = 1\cdot b_1 + 2\cdot b_2x + 3\cdot b_3x^2 + 4\cdot b_4x^3 + \ldots$$
$$f''(x) = f^{(2)}(x) = 2\cdot 1\cdot b_2 + 3\cdot 2\cdot b_3x + 4\cdot 3\cdot b_4x^2 + 5\cdot 4\cdot b_5x^3 + \ldots$$
$$f^{(3)}(x) = 3\cdot 2\cdot 1\cdot b_3 + 4\cdot 3\cdot 2\cdot b_4x + 5\cdot 4\cdot 3\cdot b_5x^3 + 6\cdot 5\cdot 4\cdot b_6x^3 + \ldots$$

and so on

$$f^{(k)}(x) = k!\,b_k + x\,g(x) \quad \text{for some power series } g(x)$$

So we want to show that $f^{(k)}(0)$ is an odd integer for all k. Let us compute some $f^{(k)}(x)$ in terms of the original $P(x)$ to see what we are up against.

$$f(x) = f^{(0)}(x) = e^{P(x)}$$
$$f'(x) = f^{(1)}(x) = e^{P(x)}P'(x)$$
$$f''(x) = f^{(2)}(x) = e^{P(x)}(P'(x)^2 + P''(x))$$

All $f^{(k)}(x)$ are of form $e^{P(x)}h_k(x)$ for some polynomial $h_k(x)$. Three observations. First, $e^{P(0)} = 1$, so $f^{(k)}(0) = h_k(0)$. Second, $P'(0) = a_1$. Third, since the second derivative of each monomial ax^m equals 0 or $m(m-1)ax^{m-2}$ with $m(m-1)$ even, we have that for each integer coefficient polynomial $p(x)$ there is an integer coefficient polynomial $q(x)$ such that $p''(x) = 2q(x)$.

We claim that each $f^{(k)}(x)$ can be written as $f^{(k)}(x) = e^{P(x)}(P'(x)^k + 2q_k(x))$ for some integer coefficient polynomial $q_k(x)$. For a proof by induction on k, we already checked the cases $k = 0, 1, 2$. Induction step: Suppose $f^{(k)}(x) = e^{P(x)}(P'(x)^k + 2q_k(x))$. Then $f^{(k+1)}(x) = e^{P(x)}P'(x)(P'(x)^k + 2q_k(x)) + e^{P(x)}(kP'(x)^{k-1}P''(x) + 2q_k'(x)) = e^{P(x)}(P'(x)^{k+1} + 2q_{k+1}(x))$ with $2q_{k+1}(x) = 2P'(x)q_k(x) + kP'(x)^{k-1}P''(x) + 2q_k'(x)$. By induction the claim holds.

So $f^{(k)}(0) = a_1^k + 2q_k(0) \equiv 1 \bmod 2$. Thus all $k!\,b_k$ are odd integers.