The complex plane

Not all polynomial functions over the real numbers $\mathbb{R}$ have a real root. The standard example is the function $f(x) = x^2 + 1$. If i were such that $f(i) = i^2 + 1 = 0$, then $i = \sqrt{-1}$ or so, an obvious absurdity for real numbers. Remarkably, we can construct a larger class of so-called *complex numbers* $\mathbb{C}$ which contains $\mathbb{R}$ as well as such a number i. This is not at all automatic. In general we cannot just extend a structure by adding a new element, and expect something nice to happen.

Imagine the collection $\mathbb{C}$ as a plane of numbers, rather than a line as for $\mathbb{R}$. Each complex number can be written as (a, b), for real numbers a and b. Addition and subtraction are as for vectors. Multiplication works like

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

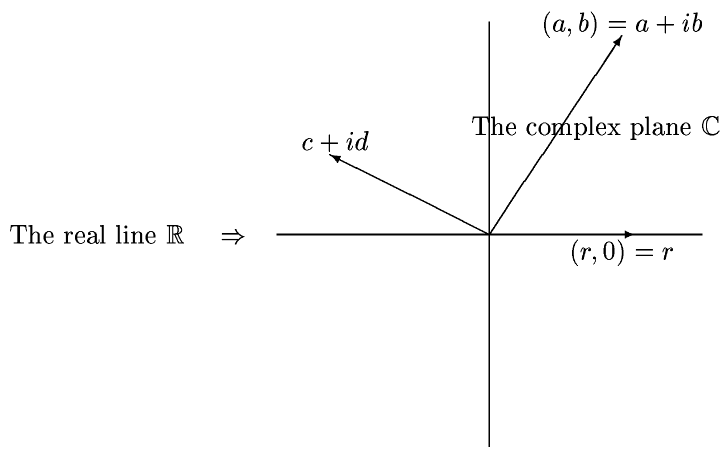
This is unpleasant to memorize. So instead of $z = (a, b)$ we usually write $z = a + ib$, where i is a reserved symbol. When we perform the multiplication above again, we get

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

We can liberally write $(ac - bd) + i(ad + bc) = ac - bd + iad + ibc$ without causing serious confusion. Now this new product looks just like an ordinary multiplication, with the additional rule that $i^2 = -1$. Remarkably, the familiar formulas of algebra still hold. We even have inverses: If $a + ib$ is nonzero (that is, at least one of a and b is nonzero), then

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$$

Finally, we identify the vectors $(r, 0)$ on the horizontal axis with the real numbers r. Note that all the algebraic manipulations of the complex numbers exactly agree with the usual real number manipulations.



We can visualize multiplication of complex (vector) numbers by multiplying the lengths of the original vectors to get the new length, and by adding the angles that the original vectors make relative to the positive x-axis, to get the angle direction of the new (vector) number relative to the positive x-axis.

The reals versus the complexes

One of the most obvious properties of real numbers $\mathbb{R}$ not shared by the complex numbers $\mathbb{C}$ is, that real numbers can be linearly ordered from left to right by $r < s$ when r is to the left (is smaller) than s. For complex numbers $z = a + ib$ we only have the 'absolute value' real valued function $|z|$ defined by $|z| = |a + ib| = \sqrt{a^2 + b^2}$, which satisfies the weaker properties

A1 $|z| \geq 0$

A2 $|z| = 0$ exactly when $z = 0$

A3 $|zw| = |z|\,|w|$

A4 $|z + w| \leq |z| + |w|$ (triangle inequality)

The value $|z|$ equals the (Euclidean) distance from the vector point z to the origin of the complex plane.

Complex numbers $\mathbb{C}$ have their advantages over $\mathbb{R}$. First recall some properties of $\mathbb{R}$: By the Intermediate Value Theorem (IVT), when the graph of a continuous function 'crosses' the horizontal axis, the function must have a real root somewhere in between. For example, the function $f(x) = x^2 - 3$ is such that $f(1) = -2$, and $f(2) = 1$. So, by the IVT, there is a real number r between 1 and 2 such that $f(r) = 0$; in fact $r = \sqrt{3}$ works.

More generally, if a is a positive real number, then $g(x) = x^2 - a$ is such that

$$g(0) = -a < 0 \quad \text{and} \quad g(\frac{a}{2} + 1) = \frac{a^2}{4} + 1 > 0$$

So, by the IVT, $g(r) = 0$ for some r between 0 and $\frac{a}{2} + 1$. As before, $r = \sqrt{a}$ works.

Here is a different application of the IVT: An odd degree polynomial function

$$h(x) = x^{2n+1} + a_{2n}x^{2n} + \ldots + a_1 x + a_0$$

turns negative when x is 'very negative', and $h(x)$ turns positive when x is 'very positive.' So $h(x)$ has a root somewhere in between. Thus all odd degree polynomials have a root in $\mathbb{R}$.

This is about as far as one can get with $\mathbb{R}$. Over the complex numbers $\mathbb{C}$, one can show that *all* polynomial functions of positive degree

$$f(z) = z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0$$

have a root in $\mathbb{C}$. This still holds when we permit the coefficients a_j to be complex numbers themselves.