Solving the one dimensional linear differential equation

A one dimensional linear differential equation is of the form

$$\frac{d}{dt}(x(t)) = a(t)x(t) + b(t) \qquad x(t_0) = x_0$$

where the functions $a(t)$ and $b(t)$ are continuous in a neighborhood of t_0. To shorten notation, one also writes

$$\dot{x} = ax + b \qquad x(t_0) = x_0$$

when it is understood that we are dealing with functions in the variable t. We obtain the unique solution as follows: Rewrite the equation as

$$\frac{d}{dt}(x(t)) - a(t)x(t) = b(t)$$

Multiply both sides by $e^{-\int_{t_0}^{t} a(s)\,ds}$ to get

$$\frac{d}{dt}(x(t))e^{-\int_{t_0}^{t} a(s)\,ds} - a(t)x(t)e^{-\int_{t_0}^{t} a(s)\,ds} = b(t)e^{-\int_{t_0}^{t} a(s)\,ds}$$

The left hand side can now be written as a derivative, so that

$$\frac{d}{dt}(x(t)e^{-\int_{t_0}^{t} a(s)\,ds}) = b(t)e^{-\int_{t_0}^{t} a(s)\,ds}$$

Integrate both sides from t_0 to t:

$$\int_{t_0}^{t} \frac{d}{du}(x(u)e^{-\int_{t_0}^{u} a(s)\,ds})\,du = \int_{t_0}^{t} b(u)e^{-\int_{t_0}^{u} a(s)\,ds}\,du$$

Evaluation of the integral of a derivative gives

$$x(t)e^{-\int_{t_0}^{t} a(s)\,ds} - x(t_0)e^{-\int_{t_0}^{t_0} a(s)\,ds} = \int_{t_0}^{t} b(u)e^{-\int_{t_0}^{u} a(s)\,ds}\,du$$

Since $\int_{t_0}^{t_0} a(s)\,ds = 0$, this reduces to

$$x(t)e^{-\int_{t_0}^{t} a(s)\,ds} - x(t_0) = \int_{t_0}^{t} b(u)e^{-\int_{t_0}^{u} a(s)\,ds}\,du$$

And thus

$$x(t) = e^{\int_{t_0}^{t} a(s)\,ds}\int_{t_0}^{t} b(u)e^{-\int_{t_0}^{u} a(s)\,ds}\,du + x(t_0)e^{\int_{t_0}^{t} a(s)\,ds}$$