Below, we use boldface letters to represent vectors. For example, 0 is the number zero, and $\mathbf{0}$ is the vector of length zero.

What is a norm on a vector space?

A *norm* on a vector space V is a function from V to the real numbers $\mathbb{R}$, usually written like $||\mathbf{v}||$, satisfying

A1 $||\mathbf{v}|| \geq 0$

A2 $||\mathbf{v}|| = 0$ exactly when $\mathbf{v} = \mathbf{0}$

A3 $||r\mathbf{v}|| = |r|\ ||\mathbf{v}||$

A4 $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$ (triangle inequality)

A norm $||\mathbf{v}||$ represents the size, or length, of a vector $\mathbf{v}$. The 'distance' between two vectors $\mathbf{v}$ and $\mathbf{w}$ is then given by the amount $||\mathbf{v} - \mathbf{w}||$.

An example: Let V be a finite dimensional vector space of the form $\mathbb{R}^n$, for some positive integer n, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector. Define $||\mathbf{x}||_a$ by

$$||\mathbf{x}||_a = |x_1| + |x_2| + \ldots + |x_n|$$

One easily verifies that this defines a norm on $V = \mathbb{R}^n$. The distance $||\mathbf{v} - \mathbf{w}||_a$ corresponds with the taxidriver distance, where the shortest path must be built from straight line segments parallel or perpendicular to all major axes.

Another example: Let V be a finite dimensional vector space of the form $\mathbb{R}^n$, for some positive integer n, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector. Define $||\mathbf{x}||_e$ by

$$||\mathbf{x}||_e = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

The function $||\mathbf{x}||_e$ satisfies all the required axioms of a norm, but it is not immediately obvious how to prove the triangle inequality. Its standard proof involves the so-called Cauchy-Schwarz inequality. The distance $||\mathbf{v} - \mathbf{w}||_e$ corresponds with the distance as the crow flies.

From the two examples above we see that a vector space can have different norms. Fortunately, on $V = \mathbb{R}^n$ they can not be too different. For example, the two norms above are bound by

$$||\mathbf{v}||_a \leq \sqrt{n}||\mathbf{v}||_e$$

and

$$||\mathbf{v}||_e \leq ||\mathbf{v}||_a$$